AN IMPROVED CONVOLUTION ALGORITHM FOR DISCRETELY SAMPLED ASIAN OPTIONS

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ABSTRACT. We suggest an improved FFT pricing algorithm for discretely sampled Asian options with general independently distributed returns in the underlying. Our work complements the studies of Carverhill and Clewlow (1990), Benhamou (2002), and Fusai and Meucci (2008), and, if we restrict our attention only to lognormally distributed returns, also Večeř (2002). While the existing convolution algorithms compute the *density* of the underlying state variable by moving *forward* on a suitably defined state space grid our new algorithm uses *backward price convolution,* which resembles classical lattice pricing algorithms. For the first time in the literature we provide an analytical upper bound for the pricing error caused by the truncation of the state space grid and by the curtailment of the integration range. We highlight the benefits of the new scheme and benchmark its performance against existing finite difference, Monte Carlo, and forward density convolution algorithms.

1. INTRODUCTION

Asian options represent a class of derivative securities whose payoff depends on an average price of the underlying asset during a prespecified time window. Their appeal stems partly from the fact that average price is harder to manipulate by unscrupulous traders and partly because the averaging reduces the volatility of the underlying asset and thus leads to lower option prices.

The manner of averaging has a significant impact on the analytical tractability of Asian options. The literature distinguishes geometric versus arithmetic averaging and continuous versus discrete sampling of the underlying price. While the geometric average admits a closed-form solution in the

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Black-Scholes model (cf. Turnbull and Wakeman 1991), for the more prevalent arithmetic average no simple analytical solution exists[1]. The Asian case is complicated by the fact that one wishes to achieve a reduction in the number of state variables. This reduction, foreshadowed in Ingersoll (1987) and employed in Rogers and Shi (1995), Andreasen (1998), Večeř (2001) and Večeř (2002), uses a change of measure and requires the option pay-off to be a homogeneous function of the weighted average of stock prices. The same applies to the PIDE of Večeř and Xu (2004) which is implemented numerically for continuously monitored Asians under jump-diffusions in Bayraktar and Xing (2008).

Most Asian options are not sampled continuously, indeed it is typical for the underlying price to be recorded only once a day or once a week. The above PDE techniques[2] can be modified to accommodate discrete sampling (cf. Andreasen 1998, Večeř 2002), but they do not exploit discrete sampling to their advantage. If anything, discrete sampling has a detrimental effect on the finite difference algorithms by making the PDE coefficients discontinuous and therefore eroding the quadratic convergence in time of the Crank-Nicolson scheme.

There are three existing papers, Carverhill and Clewlow (1990), Benhamou (2002) and Fusai and Meucci (2008), which are specifically tailored to discretely sampled Asians while simultaneously exploiting a state space reduction similar to the one mentioned above, using so-called Carverhill-Clewlow-Hodges factorization. In all three papers the computation works by evaluating the *density* of the average price going *forward*. These papers have a significant advantage over the PDE techniques in that they are easily extended to allow for leptokurtic returns in the underlying without being restricted to jump-diffusions.

By exploiting the notion of a *reverse filtration* we are able to replace the Carverhill-Clewlow *forward density* convolution by a *backward price* convolution. We demonstrate below that this has substantial numerical and theoretical advantages. The paper is organized as follows: In Section 2 we introduce the notation and the Carverhill-Clewlow-Hodges factorization. In Section 3 we develop the main theoretical results for the backward price

[1]See Linetsky (2004) and references therein for transform methods and their numerical implementation.

[2]All the above PDEs, apart from Večeř (2002), suffer from instability under standard (explicit, implicit or Crank-Nicolson) finite difference schemes. The instability occurs because the drift may dominate the diffusion coefficient in some regions of the grid. The instability can be remedied by "upwind" differences, at the cost of lower convergence speed in spatial dimension (cf. Kushner and Dupuis 2001). Therefore, of the PDE schemes surveyed here, only Večeř (2002) is numerically competitive.

convolution scheme, and in Section 4 we discuss its implementation via discrete Fourier transform. In Section 5 we describe parametrization of log return distribution for the Black-Scholes model and two Lévy process models, the tempered stable and normal inverse Gaussian. Section 6 concludes by comparing the speed and accuracy of our scheme with previous studies.

2. Preliminaries

Fix $n \in \mathbb{N}$, and let $\{Z_k\}_{k=1}^n$ be a collection of independent random variables on the probability space $\{\Omega, \mathcal{F}, P\}$ such that $0 < \mathrm{Var}(\exp(Z_k)) < \infty$. Let $\mathbb{F} := \{\mathcal{F}_k\}_{k=1}^n$ be the information filtration generated by the random variables $\{Z_k\}$, with $\mathcal{F}_0$ trivial. Fix a constant $S_0 > 0$ and define the price process of a risky asset

$$S_j := S_0 \exp(\sum_{k=1}^{j} Z_k),\ j = 1, \ldots, n,$$

and a risk-free bank account with total return R_k in period $k \in \{1, \ldots, n\}$. We assume period k dividend payment of the size $(D_k - 1)\, S_{k-1}, D_k \geq 1$.

We interpret P as a risk-neutral measure. In the presence of dividends we therefore have

$$E(e^{Z_k}) = R_k / D_k := \mu_k \text{ for } k = 1, \ldots, n.$$

The collection $\{Z_k\}_{k=1}^n$ is completely general at this stage. In specific applications one may allow Z_k-s to follow a non-parametric distribution or identify them with increments of a specific Lévy process (see Section 5). In the Black–Scholes model with sampling dates $t_1, \ldots, t_n$ one has

$$\begin{aligned} Z_k &\sim N((r - \sigma^2/2)(t_k - t_{k-1}), \sigma^2(t_k - t_{k-1})), \\ R_k &= \exp(r(t_k - t_{k-1})), \\ D_k &= \exp(\hat{\delta}(t_k - t_{k-1})), \end{aligned}$$

where σ represents the volatility of log returns, r is the risk-free rate and $\hat{\delta}$ the dividend yield.

Fix a deterministic process λ and define a process of partial sums

$$I_j := \sum_{k=0}^{j} \lambda_k S_k. \tag{2.1}$$

Pricing of fixed or floating strike call/put options amounts to calculating

$$E(I_n^+), \tag{2.2}$$

for different choices of process λ (this unification is described in Večeř 2002) shown in Table 1. With appropriate choice of λ one can also capture forward start options and in-progress options.

Option type	λ_0	$\lambda_1, \ldots, \lambda_{n-1}$	λ_n
Call, fixed strike	$\frac{\gamma}{n+\gamma} - \frac{K}{S_0}$	$\frac{1}{n+\gamma}$	$\frac{1}{n+\gamma}$
Call, floating strike	$-\frac{\gamma\alpha}{n+\gamma}$	$-\frac{\alpha}{n+\gamma}$	$1 - \frac{\alpha}{n+\gamma}$
Put, fixed strike	$\frac{K}{S_0} - \frac{\gamma}{n+\gamma}$	$-\frac{1}{n+\gamma}$	$-\frac{1}{n+\gamma}$
Put, floating strike	$\frac{\gamma\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma} - 1$

TABLE 1. Choice of λ corresponding to different types of Asian options. $\alpha > 0$ is the coefficient of partiality for floating strike options. Coefficient γ takes value 1 (0) when S_0 is (is not) included in the average.

The computational difficulty stems from the fact that I is not a Markov process under P. More specifically S is Markov and (S, I) are jointly Markov under P which means, when evaluating (2.2) recursively, that the conditional expectation $E(I_n^+|\mathcal{F}_t)$ depends on both S_t and I_t. This implies pricing must be performed on a 3-dimensional grid (I, S, time).

Let us now define a new filtration $\mathbb{G} := \{\mathcal{G}_i\}_{i=1}^n$

$$\mathcal{G}_i = \sigma\{Z_n, Z_{n-1}, \ldots, Z_{n-i+1}\},$$

and process X by setting

$$\begin{aligned} X_k &:= \lambda_{n-k} + X_{k-1} \exp(Z_{n+1-k}), \\ X_0 &:= \lambda_n. \end{aligned} \tag{2.3}$$

We can now state a generalized version of the Carverhill-Clewlow-Hodges factorization[3].

Proposition 2.1. *Consider deterministic coefficients $\{\lambda_k\}_{k=0}^n$ and processes I, X defined in (2.1), (2.3). The following statements hold:*

(1) *The random variables I_n, X_n satisfy*

$$I_n = S_0 X_n; \tag{2.4}$$

(2) *Process X is Markov in filtration $\mathbb{G}$ under measure P.*

Proof. (1) follows by recursive substitution. (2) follows from the stochastic independence of variables $\{Z_k\}_{k=1}^n$. □

[3]We do not require λ to be constant or of the same sign and we do not need variables Z to be identically distributed.

3. Backward price convolution algorithm

Proposition 2.1 signifies that one can price the Asian option by evaluating $S_0E(X_n^+)$ recursively in filtration $\mathbb{G}$. Now assume that $\lambda_i \geq 0$ for $i = 1, \ldots, n$. This, by virtue of (2.3), implies $X_k > 0$ for $0 \leq k < n$ and in turn we obtain

$$X_k - \lambda_{n-k} > 0 \text{ for } 1 \leq k < n.$$

Thus we can write (2.3) alternatively as

$$\ln(X_k - \lambda_{n-k}) = \ln X_{k-1} + Z_{n+1-k}. \tag{3.1}$$

Carverhill and Clewlow (1990), Benhamou (2002) and Fusai and Meucci (2008) use this transition equation to compute the unconditional risk-neutral density of $\ln(X_n - \lambda_0)$ which they subsequently use to price a fixed strike Asian call option. This is done recursively, evaluating the density of $\ln(X_k - \lambda_{n-k})$ as the convolution of densities of $\ln X_{k-1}$ and Z_{n+1-k} in line with equation (3.1). In the first two papers the convolution is computed by Fourier transform, in the third paper it is computed directly.

In all three papers the difficulty is that the density of $\ln(X_k - \lambda_{n-k})$ spreads out as k increases. This problem is not specific to Asian options, the same situation would occur if one applied the density method iteratively to plain vanilla options in the Black-Scholes model. Ideally, one should use a dense and narrow grid for $\ln(X_1 - \lambda_{n-1})$ and wide and relatively sparse grid for $\ln(X_n - \lambda_0)$. Clewlow and Carverhill use the same equidistantly spaced grid for all variables $\ln(X_k - \lambda_{n-k})$. Benhamou (2002) models *re-centered* variables $\ln(X_k - \lambda_{n-k})$ on a common grid. Fusai and Meucci (2008) use a fixed grid that is not equidistant to take advantage of Gaussian numerical integration. Simultaneous recentering and rescaling has not been implemented in the literature, to the best of our knowledge, although the methodology of Fusai and Meucci (2008) allows this in principle.

The main difference in our approach to the foregoing papers is that we model the price of the Asian option directly, not the density of the underlying average. This allows us to bypass the computation of the density altogether. Our method thus resembles backward pricing in a lattice, as opposed to forward density convolution. In the next theorem we write down our recursive pricing algorithm together with some price bounds needed later. Note that smoothness of densities $\{f_k\}_{k=1}^n$ is not required, therefore our result is applicable also to models where the density has a singularity. This occurs, for example, in the variance-gamma or CGMY models at very short time horizons.

Theorem 3.1. *Assume that for all k the CDF of Z_{n+1-k} has a density f_k with respect to the Lebesgue measure on $\mathbb{R}$, satisfying*

$$\mu_k := \int_{\mathbb{R}} e^z f_k(z)dz < \infty.$$

Consider constants $\lambda_k > 0, 0 < k \leq n$ and $\lambda_0 \in \mathbb{R}$. Define functions $p_k : \mathbb{R} \to \mathbb{R}$ for $0 < k \leq n$ and $q_k, h_k : \mathbb{R} \to \mathbb{R}$ for $0 \leq k < n$ as follows

$$\begin{aligned} p_n(y) &:= (e^y + \lambda_0)^+, \\ h_k(y) &:= \ln(e^y + \lambda_{n-k}), 0 < k < n, \\ q_{k-1}(x) &:= \int_{\mathbb{R}} p_k(x+z) f_k(z)dz, 0 < k \leq n, \qquad (3.2) \\ p_{k-1}(y) &:= q_{k-1}(h_{k-1}(y)), 1 < k \leq n. \qquad (3.3) \end{aligned}$$

The following statements hold:

(1) *The forward price of an Asian call contract with parameters $\{\lambda_j\}_{j=0}^n$ is given by*

$$E(I_n^+) = S_0 E(X_n^+) = S_0 q_0(\ln \lambda_n).$$

(2) *There are positive constants a_k, b_k such that for all $x, y \in \mathbb{R}$*

$$\begin{aligned} 0 &\leq p_k(y) \leq a_k e^y + b_k, \qquad (3.4) \\ 0 &\leq q_k(x) \leq a_k e^x + b_{k+1}. \qquad (3.5) \end{aligned}$$

These constants are given recursively by

$$\begin{aligned} a_n &= 1, b_n = \lambda_0^+, \qquad (3.6) \\ a_{k-1} &= a_k \mu_k, \qquad (3.7) \\ b_{k-1} &= b_k + a_{k-1}\lambda_{n-k+1}. \qquad (3.8) \end{aligned}$$

In numerical applications we must curtail the range of integration in (3.2). In the next theorem we estimate the pricing error caused by the curtailment and provide a constructive method for finding the curtailed ranges.

Theorem 3.2. *Consider the following (exponential) tail moments*

$$\begin{aligned} F_k(z;\beta) &= \int_{-\infty}^{z} e^{\beta x} f_k(x)dx, \\ G_k(z;\beta) &= \int_{z}^{\infty} e^{\beta x} f_k(x)dx. \end{aligned}$$

Under the assumptions of Theorem 3.1 $F_k(z;1) + G_k(z;1) = \mu_k$ and

$$\lim_{z\to-\infty} F_k(z;\beta) = \lim_{z\to\infty} G_k(z;\beta) = 0 \text{ for } \beta = 0, 1.$$

Consider compact intervals $\{[l_k, u_k]\}_{k=1}^n$ *and define compact intervals* $\{[L_k, U_k]\}_{k=1}^n$, $\{[\bar{L}_k, \bar{U}_k]\}_{k=0}^{n-1}$ *by setting*

$$\begin{aligned} \bar{L}_0 &= \bar{U}_0 = \ln \lambda_n, && (3.9)\\ L_k &= \bar{L}_{k-1} + l_k, U_k = \bar{U}_{k-1} + u_k, 0 < k \leq n, && (3.10)\\ \bar{L}_k &= \ln(e^{L_k} + \lambda_{n-k}), \bar{U}_k = \ln(e^{U_k} + \lambda_{n-k}), 0 < k < n. && (3.11) \end{aligned}$$

Define functions $\tilde{q}_k, \tilde{p}_k$ *by setting*

$$\begin{aligned} \tilde{q}_{k-1}(x) &:= (\int_{\mathbb{R}} \tilde{p}_k(x+z) f_k(z) dz) 1_{[\bar{L}_{k-1}, \bar{U}_{k-1}]}(x), 0 < k \leq n,\\ \tilde{p}_{k-1}(y) &:= \tilde{q}_{k-1}(h_{k-1}(y)) 1_{[L_{k-1}, U_{k-1}]}(y), 1 < k \leq n,\\ \tilde{p}_n(y) &:= p_n(y) 1_{[L_n, U_n]}(y) \end{aligned}$$

Let $\tilde{a}_n = \tilde{b}_n = 0$ *and for* $0 < k \leq n$ *recursively define*

$$\begin{aligned} \tilde{a}_{k-1} &= \tilde{a}_k \mu_k + a_k(F_k(l_k; 1) + G_k(u_k; 1)),\\ \tilde{b}_{k-1} &= \tilde{a}_{k-1} \lambda_{n-k+1} + \tilde{b}_k + b_k(F_k(l_k; 0) + G_k(l_k; 0)), \end{aligned}$$

where a_k, b_k *are given by (3.6-3.8).*

(1) *The pricing error has the following bounds*

$$0 \leq q_0(\ln \lambda_n) - \tilde{q}_0(\ln \lambda_n) \leq \tilde{b}_0, \quad (3.12)$$

and $\tilde{b}_0 > 0$ *can be made arbitrarily small by suitable choice of* $\{[l_k, u_k]\}_{k=1}^n$.

(2) *For* $k < n$ *and positive constants* c_k *defined recursively by* $c_n = 0, c_{k-1} = c_k + a_{k-1}\lambda_{n-k+1}$ *we have*

$$\begin{aligned} 0 &\leq \tilde{p}_k'(y) \leq a_k e^y + c_k \text{ for } y \in (L_k, U_k), && (3.13)\\ 0 &\leq \tilde{q}_k'(x) \leq a_{k-1} e^x + c_k \text{ for } x \in (\bar{L}_k, \bar{U}_k) && (3.14) \end{aligned}$$

(3) *Functions* $\{\tilde{p}_k\}_{k=1}^{n-1}, \{\tilde{q}_k\}_{k=0}^{n-1}$ *are continuously differentiable on the interior of their support.*

(4) *For* $0 \leq k < n$ *and* $x \in (\bar{L}_k, \bar{U}_k)$

$$\tilde{q}_{k-1}(x) = \mathcal{F}^{-1}(\mathcal{F}(\tilde{p}_k)\bar{\phi}_k)(x), \quad (3.15)$$

where $\mathcal{F}$ *denotes the Fourier transform (see Appendix A),* ϕ_k *is the characteristic function of* Z_{n-k}

$$\phi_k(u) := \int_{\mathbb{R}} e^{iuz} f_k(z) dz,$$

and $\bar{\phi}_k$ *denotes its complex conjugate.*

It is straightforward to modify Theorem 3.1 to obtain price sensitivities (greeks). This involves taking the derivative of the price with respect to the parameter of interest, which amounts, roughly speaking, to interchanging the order of integration and differentiation in equation (3.2). The same

principle is used in Lord et al. (2008) for the computation of the delta and gamma of a European plain vanilla option, and more generally in the case of Bermudan options with multiple exercise dates. Using this method the computational effort for sensitivities is of the same order as computational effort for prices.

4. Numerical implementation

Definition 4.1. *Consider two uniform grids* $\mathbf{x} = \{x_0 + j\delta x\}_{j=0}^{n-1}$ *and* $\mathbf{u} = \{u_0 + k\delta u\}_{k=0}^{m-1}$ *and vector* $a = \{a_j\}_{j=0}^{n-1}$. *We define a generalized discrete Fourier transform (DFT) of a from* $\mathbf{x}$ *onto* $\mathbf{u}$ *as the* m*-dimensional vector* $b = \{b_k\}_{k=0}^{m-1}$ *satisfying*

$$b_k = \sum_{j=0}^{n-1} a_j e^{i\mathbf{x}_j \mathbf{u}_k} = \sum_{j=0}^{n-1} a_j e^{i(x_0+j\delta x)(u_0+k\delta u)}.$$

We write $b = \mathcal{D}(a, \mathbf{x}, \mathbf{u})$.

This is a very slight generalization of transforms that feature prominently in the signal processing literature. When $x_0 = u_0 = 0$ we obtain so-called fractional Fourier transform with fractionality coefficient $\alpha = \frac{n\delta x \delta u}{2\pi}$. When furthermore $m = n$ and $\alpha = 1$ we obtain the standard DFT. In this paper we compute the generalized DFT by means of so-called chirp z-transform[4] (CZT), which is readily available in MATLAB.

Definition 4.2. *Consider vector* $a = \{a_j\}_{j=0}^{n-1}$, *and parameters* $m \in \mathbb{N}$, $A, w \in \mathbb{C}$. *The chirp* z*-transform of* a *with parameters* A, w, m *is the vector* $b = \{b_k\}_{k=0}^{m-1}$ *satisfying*

$$b_k = \sum_{j=0}^{n-1} a_j (Aw^{-k})^j.$$

We write $b = \mathrm{czt}(a, A, w, m)$.

Proposition 4.3. *Consider two uniform grids* $\mathbf{x} = \{x_0 + j\delta x\}_{j=0}^{n-1}$ *and* $\mathbf{u} = \{u_0 + k\delta u\}_{k=0}^{m-1}$ *and vector* $a = \{a_j\}_{j=0}^{n-1}$. *We have*

$$\mathcal{D}(a, \mathbf{x}, \mathbf{u}) = e^{ix_0\mathbf{u}} \mathrm{czt}(a, e^{iu_0\delta x}, e^{-i\delta x \delta u}, m). \tag{4.1}$$

Proof. Straightforward algebra. □

Our numerical algorithm proceeds as follows:

[4]CZT pre-dates FrFT by more than 20 years, see Rabiner et al. (1969). It is more general in form but has the same computational cost. It can be evaluated at the cost of 3 standard DFTs, using so-called Bluestein (1968) algorithm. For more details on the speed and implementation in the context of option pricing see Černý (2004).

(1) For a given contract we find values of l_k, u_k that achieve a predetermined pricing precision given by (3.12). We evaluate the tail moments by Fourier inversion

$$\begin{aligned} F_k(x;\beta) &= -\frac{1}{2\pi}\lim_{c\to\infty}\int_{-c}^{c}\frac{\phi_k(u-i(\beta+\zeta))}{iu+\zeta}e^{-(iu+\zeta)x}du, \\ G_k(x;\beta) &= \frac{1}{2\pi}\lim_{c\to\infty}\int_{-c}^{c}\frac{\phi_k(u-i(\beta+\eta))}{iu+\eta}e^{-(iu+\eta)x}du, \end{aligned}$$

for suitably chosen constants $\zeta < 0, \eta > 0$ (cf. Strawderman 2004).

(2) We then determine the grid ranges $[L_k, U_k]$ and $[\bar{L}_k, \bar{U}_k]$ for functions $\tilde{p}_k, \tilde{q}_k$ via (3.9-3.11).

(3) We select a uniform grid $\mathbf{u}$ symmetric around zero. The range of values in $\mathbf{u}$ is determined to ensure $|\phi_k| < \rho$ outside $\mathbf{u}$. The value of ρ is guided by the desired precision, i.e. we pick $\rho = 10^{-7}$ when computing results to 7 decimal places.

(4) Suppose the values approximating $\tilde{p}_k$ are given on a uniform grid $\mathbf{y}$ and the values of ϕ_k are given on a uniform grid $\mathbf{u}$. By abuse of notation we denote the function values on the grid by $\tilde{\mathbf{p}}_k$ and $\boldsymbol{\phi}_k$, respectively. We evaluate $\mathbf{P}_k := \mathcal{D}(\tilde{\mathbf{p}}_k, \mathbf{y}, \mathbf{u})$ as a discrete approximation of the transform $\mathcal{F}(\tilde{p}_k)$. We then approximate the inverse Fourier transform (3.15) by computing $\tilde{\mathbf{q}}_{k-1} = \frac{\mathcal{D}(\mathbf{P}_k\bar{\boldsymbol{\phi}}_k, -\mathbf{u}, \mathbf{x})}{2\pi}$ where $\mathbf{x}$ is a uniform grid on the interval $[\bar{L}_{k-1}, \bar{U}_{k-1}]$. The generalized DFT is implemented via fast CZT using the conversion (4.1).

(5) We approximate $\tilde{q}_{k-1}$ inside grid $\mathbf{x}$ by a cubic interpolating spline fitted to the nodes $(\mathbf{x}, \tilde{\mathbf{q}}_{k-1})$ using not-a-knot endpoint conditions. Outside $\mathbf{x}$ we approximate $\tilde{q}_{k-1}$ by extrapolating linearly in e^x. We compute $\tilde{\mathbf{p}}_{k-1} = \tilde{q}_{k-1}(h_{k-1}(\mathbf{y}))$ and continue with item 4.

Theoretically we expect quadratic convergence of the price as a function of the grid spacing for grids $\mathbf{x}, \mathbf{y}, \mathbf{u}$. Let α_n be a numerical price corresponding to multiplier 2^{-n} of the original spacing on grids $\mathbf{x}, \mathbf{y}, \mathbf{u}$. We expect $\alpha_n - \alpha_{n+1} \approx 4(\alpha_{n+1} - \alpha_{n+2})$ and we indeed observe $\alpha_n - \alpha_{n+1} = (4 \pm 10^{-2})(\alpha_{n+1} - \alpha_{n+2})$ for n sufficiently high. At this point if two consecutive Richardson extrapolations agree to the desired precision we terminate the algorithm. We then increase grid range for $\mathbf{u}$ and run the whole computation again to confirm that the results agree to the desired number of decimal places.

5. Leptokurtic stock returns

We evaluate option prices numerically based on three distributions of log returns: normal (Gaussian), tempered stable (CGMY) and normal inverse Gaussian (NIG). All three are infinitely divisible and therefore consistent with a continuous-time Lévy model for the log stock price. Denoting the

underlying Lévy process by L the risk-neutral characteristic function ϕ is given by the formula

$$\begin{aligned} E(e^{iuL_t}) &= e^{\hat{\kappa}(iu)t}, \\ \hat{\kappa}(u) &= \kappa(u) + u(r - \hat{\delta} - \kappa(1)), \end{aligned} \tag{5.1}$$

where t is the time horizon measured in years and $\hat{\delta}$ is the dividend yield. The cumulant generating functions $\kappa(u)$ for the different models are given as follows,

$$\begin{aligned} \kappa_{\mathrm{G}}(u) &= \sigma^2 u^2/2, \\ \kappa_{\mathrm{CGMY}}(u) &= C\Gamma(-Y)\left((M-u)^Y - M^Y + (G+u)^Y - G^Y\right), \\ \kappa_{\mathrm{NIG}}(u) &= (1 - \sqrt{1 - 2\theta\nu u - \nu\sigma^2 u^2})/\nu. \end{aligned}$$

By a standard result (cf. Cont and Tankov 2004, Section 2.2.5) L has the following unconditional moments

$$\begin{aligned} E(L_t) &= \hat{\kappa}'(0)t = (\kappa'(0) + r - \hat{\delta} - \kappa(1))t, \\ \mathrm{Var}(L_t) &= \hat{\kappa}''(0)t = \kappa''(0)t, \\ \mathrm{skew}(L_t) &= \frac{\hat{\kappa}'''(0)}{(\hat{\kappa}''(0))^{3/2}} t^{-1/2} = \frac{\kappa'''(0)}{(\kappa''(0))^{3/2}} t^{-1/2}, \\ \mathrm{kurt}(L_t) &= 3 + \frac{\hat{\kappa}^{(4)}(0)}{(\hat{\kappa}''(0))^2} t^{-1} = 3 + \frac{\kappa^{(4)}(0)}{(\kappa''(0))^2} t^{-1}. \end{aligned}$$

We calibrate the three models to achieve $\sqrt{\mathrm{Var}(L_1)} \in \{0.1, 0.3, 0.5\}$ and for the non-Gaussian distributions we further require

$$\mathrm{skew}(L_1) = -0.5, \tag{5.2}$$

$$\mathrm{kurt}(L_1) = 3.7. \tag{5.3}$$

These moments are broadly consistent with risk-neutral densities fitted to option price data by Madan et al. (1998) and Carr et al. (2002). The fitted parameters, rounded to four leading digits, are summarized in Table 2.

Gaussian	NIG			tempered stable			
σ	ν	σ	θ	C	G	M	Y
0.1	0.1222	0.0879	-0.1364	0.2703	17.56	54.82	0.8
0.3	0.1222	0.2637	-0.4091	0.6509	5.853	18.27	0.8
0.5	0.1222	0.4395	-0.6819	0.9795	3.512	10.96	0.8

TABLE 2. Calibrated model parameters

6. Numerical results

6.1. **Black-Scholes model.** Our algorithm is a major improvement over existing density convolution pricing procedures. In Table 3 we present our results with precision $\pm 10^{-7}$, alongside the original numbers of Carverhill and Clewlow (1990) and Benhamou (2002). We observe that the original results are at best precise to 3 decimal places, but often not even to 2 decimal places. The precision is better for low volatilities.

vol	strike	Černý & Kyriakou	Ben-hamou	error	Carverhill & Clewlow	error
	80	22.7771749	22.7838	0.007	22.78	0.003
	90	13.7337773	13.7347	0.001	13.73	-0.004
0.1	100	5.2489927	5.2438	-0.005	5.25	0.001
	110	0.7238324	0.7211	-0.003	0.72	-0.004
	120	0.0264092	0.0336	0.007	0.02	-0.006
	80	23.0914378	23.0733	-0.018	23.09	-0.001
	90	15.2207610	15.2231	0.002	15.29	0.069
0.3	100	9.0271888	9.0110	-0.016	9.08	0.053
	110	4.8349071	4.8338	-0.001	4.86	0.025
	120	2.3682854	2.3545	-0.014	2.4	0.032
	80	24.8242581	24.8324	0.008	25.01	0.186
	90	18.3316740	18.3207	-0.011	18.5	0.168
0.5	100	13.1580456	13.1811	0.023	13.47	0.312
	110	9.2345134	9.2300	-0.005	9.45	0.215
	120	6.3719536	6.3615	-0.010	6.68	0.308

Table 3. Comparison with Benhamou (2002) and Carverhill and Clewlow (1990) for lognormal returns. Asian call option parameters $T = 1, n = 50, S_0 = 100$.

The relative lack of precision is typical for the density convolution method because one does not obtain monotone convergence in the number of grid points. This is apparent also in the most recent density convolution implementation, due to Fusai and Meucci (2008), which is in itself a substantial improvement over Carverhill and Clewlow (1990) and Benhamou (2002). In Table 4 we reproduce Fusai and Meucci's results for 1000, 5000 and 10000 quadrature points, respectively. In an ideal world the results with higher number of quadrature points should also be more precise. However, we note that the error for 1000 gridpoints is in two cases smaller than the error for 5000 and 10000 grid points. The error for 5000 points is smaller than the

error for 10000 points in further four cases. This makes it hard for users of the density convolution method to gauge the precision of their scheme. Correspondingly, one can see that the results in Fusai and Meucci can be substantially less precise than the reported 5 decimal places.

n	strike	Černý & Kyriakou	Fusai & Meucci			error $\times 10^{-5}$		
		$\pm 10^{-7}$	10000	5000	1000	10000	5000	1000
	90	11.9049157	11.90497	11.90498	11.90428	5	6	-64
12	100	4.8819616	4.88210	4.88212	4.88199	14	16	3
	110	1.3630380	1.36314	1.36314	1.36371	10	10	67
	90	11.9329382	11.93301	11.93299	11.93339	7	5	45
50	100	4.9372028	4.93736	4.93738	4.93711	16	18	-9
	110	1.4025155	1.40264	1.40262	1.40199	12	10	-53
	90	11.9405632	11.94068	11.94069	11.94137	12	13	81
250	100	4.9521569	4.95233	4.95239	4.94942	17	23	-274
	110	1.4133670	1.41351	1.41350	1.41290	14	13	-47

TABLE 4. Comparison with Fusai and Meucci (2008) for lognormal returns. Parameters $T = 1, \sigma = 0.17801, r = 0.0367, n = 50, S_0 = 100$. Numbers 1000, 5000, 10000 in the last 6 columns signify the number of grid points used by Fusai and Meucci.

The execution time is in our favour since our control variate Monte Carlo takes longer than that of Fusai and Meucci (for $n = 50$ F&M need 130 seconds to run 1,000,000 MC trials while we need 190 seconds) but our pricing procedure is substantially faster (we need under 1 second to achieve guaranteed 5 decimal place precision while F&M require 5 seconds with 1000 quadrature points to achieve a variable precision of 3-4 decimal places; see our timings in Table 7).

Monte Carlo is typically too slow to compete with lattice-based methods at low dimensions. However, in the Asian case this is not a foregone conclusion because geometric Asian options provide a control variate technique that works extremely well (cf. Kemna and Vorst 1990). In Table 5 we report standard errors and timings for the control variate method. The conclusion is that even with a very effective control variate the Monte Carlo method is not competitive as it requires 19 seconds to achieve 3 decimal places of

accuracy (at 99% confidence level) while our method needs under 1 second to obtain 5 decimal places.

n	strike	CVMC	std $\times 10^{-5}$	99% CI $\times 10^{-5}$	CPU sec
	90	11.90347	67	172	
12	100	4.88162	54	139	9
	110	1.36363	48	125	
	90	11.93295	64	165	
50	100	4.93688	53	137	19
	110	1.40213	48	125	
	90	11.94073	63	163	
250	100	4.95186	53	137	77
	110	1.41365	47	122	

TABLE 5. Results of control variate Monte Carlo method with 100,000 simulations and lognormal returns. Parameters $T = 1, \sigma = 0.17801, r = 0.0367, n = 50, S_0 = 100$. CI = confidence interval; std = standard deviation of the CVMC estimator. CPU timings are for Matlab R15 on Dell Latitude 620 Intel Dual Core T7200, 2GHz, 2Gb RAM.

The convolution methods discussed above can handle arbitrary distribution of log returns. If we restrict our attention only to lognormal returns there is another competing method: finite difference schemes[5] (cf. Večeř 2002). The pricing error of the Crank-Nicolson scheme for 50 sampling dates is shown in Table 6. Here our recursion wins easily since we can achieve 5 decimal places of precision across strikes and volatilities in under 1 second (cf. timings in Table 7). Obviously the finite difference scheme becomes more competitive as the number of sampling dates increases. We find that the break-even point is at around 200 sampling dates for precision 10^{-5} and volatility of 30%. The finite difference scheme performs better for low volatility and lower precision while our method has an extra edge for high volatility and higher precision levels.

[5]We do not use Matlab's built-in PDE solver but rather we design a customized parabolic PDE solver in which we implement an efficient sparse LU decomposition to speed up matrix inversion in the Crank-Nicolson scheme. Furthermore we extrapolate the resulting prices quadratically in time and space step to increase accuracy, which is particularly important for high volatility levels.

vol	strike				
	80	90	100	110	120
0.1	-4.5E-8	8.0E-5	8.6E-4	4.0E-4	2.1E-5
0.3	9.1E-4	2.0E-3	2.7E-3	2.5E-3	1.8E-3
0.5	2.9E-3	4.0E-3	4.4E-3	4.3E-3	3.7E-3

TABLE 6. Precision of Crank-Nicolson finite difference scheme (lognormal returns). For the construction of the pricing PDE see Večeř (2002). Spatial step (0.01,0.005); time step (1/500,1/1000). Execution time 1.1s per strike in Matlab R15 on Dell Latitude 620 Intel Dual Core T7200, 2GHz, 2Gb RAM.

6.2. **Lévy models.** We will not detail numerical comparisons with existing convolution methods for Lévy log returns. In short, the oscillatory convergence of the forward density convolution is exacerbated further by leptokurtic returns and the advantage of our method is even more pronounced than in the lognormal case.

The performance of control variate Monte Carlo strategies in the Lévy case has been examined in detail by Fusai and Meucci (2008). The main conclusion remains the same as in the lognormal model — while CVMC leads to a substantial reduction in estimation error compared to the standard MC, the scheme is nevertheless not competitive with convolution methods.

In Table 7 we report call option prices for weekly sampling, with risk-free rate 4% p.a., zero dividend yield and a year to maturity ($S_0 = 100, T = 1$, $n = 50$, $r = 0.04$, $\hat{\delta} = 0$ and $t = T/n$ in 5.1). Model parameters are given in Table 2. The precision of the reported numbers is $\pm 10^{-5}$. We can achieve higher precision (up to 10^{-8}) by exploiting the regular quadratic convergence of our scheme in the number of gridpoints. Beyond 10^{-8} smooth convergence requires too many grid points to manage the computations in reasonable time.

Two comments are in order. Comparing leptokurtic prices with lognormal prices we find in-the-money options to be slightly more expensive while out-of-the-money options are substantially cheaper. This pattern is consistent with plain vanilla option prices ($n = 1$) and is caused by a combination of excess kurtosis and negative skew in the risk-neutral distribution. The same pattern is observed by Fusai and Meucci.

Secondly, the prices generated by the two Lévy models coincide to penny accuracy. This suggests that the skewness and kurtosis of the risk-neutral

model	vol	strike			CPU
		90	100	110	sec
Gaussian	0.1	11.58113	3.33861	0.27375	1.0
	0.3	13.66981	7.69859	3.89639	0.3
	0.5	17.19239	12.09153	8.31441	0.3
NIG	0.1	11.64024	3.32385	0.15835	3.7
	0.3	13.70084	7.34265	3.27860	1.8
	0.5	16.76306	11.23586	7.16836	1.8
CGMY	0.1	11.63988	3.32458	0.15787	8.5
	0.3	13.70160	7.34742	3.28308	4.1
	0.5	16.76835	11.24424	7.17624	2.1

TABLE 7. Asian call option prices for Lévy log returns. Precision $\pm 10^{-5}$. For detailed description of parameter values see main text. CPU timings in Matlab R15 on Dell Latitude 620 Intel Dual Core T7200, 2GHz, 2Gb RAM.

distribution (cf. equations 5.2, 5.3) are the primary factors driving option prices, while the choice of a specific parametric Lévy model plays a secondary role.

APPENDIX A. FOURIER TRANSFORM

Definition A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function. The Fourier transform $\mathcal{F}(f) : \mathbb{R} \to \mathbb{C}$ is given by*

$$\mathcal{F}(f)(u) = \int_{-\infty}^{\infty} f(s) e^{ius} ds. \tag{A.1}$$

By slight abuse of notation we write $\mathcal{F}(f(s))$, even though the variable s is immaterial. For example the Fourier transform of $f(s) = g(s)e^{\alpha s}$ would be denoted $\mathcal{F}(g(s)e^{\alpha s})$.

The inverse Fourier transform of $g : \mathbb{R} \to \mathbb{C}$ is given by

$$\mathcal{F}^{-1}(g)(s) = \lim_{c \to \infty} \frac{1}{2\pi} \int_{-c}^{c} g(u) e^{-ius} du$$

whenever the limit on the right hand side exists for all $s \in \mathbb{R}$.

Theorem A.2. *Suppose f is absolutely integrable. On any compact interval where f is continuous and of finite variation the inverse Fourier transform of $\mathcal{F}(f)$ is well defined and*

$$f = \mathcal{F}^{-1}(\mathcal{F}(f)).$$

Proof. See Goldberg (1961), Theorem 5C. □

Theorem A.3. *If f_1, f_2 are absolutely integrable then*

$$(f_1 * f_2)(s) := \int_{-\infty}^{\infty} f_1(s')f_2(s-s')ds' = \int_{-\infty}^{\infty} f_2(s')f_1(s-s')ds'$$

is absolutely integrable and

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2).$$

Proof. See Goldberg (1961), Theorems 7D, 7E. □

Appendix B. Proofs

Proof of Theorem 3.1. 1) We will prove by induction on k that $E(X_n^+|\mathcal{G}_k) = p_k(\ln(X_k - \lambda_{n-k}))$ for $k = 0, \ldots, n$. The statement clearly holds for $k = n$. Assume therefore that $E(X_n^+|\mathcal{G}_{k+1}) = p_{k+1}(\ln(X_{k+1} - \lambda_{n-k-1}))$ holds. By the law of iterated expectations

$$E(X_n^+|\mathcal{G}_k) = E(E(X_n^+|\mathcal{G}_{k+1})|\mathcal{G}_k) = E(p_{k+1}(\ln(X_{k+1} - \lambda_{n-k-1}))|\mathcal{G}_k).$$

Now substitute $X_{k+1} - \lambda_{n-k-1} = X_k \exp(Z_{n-k})$ from (2.3) to obtain

$$\begin{aligned} E(X_n^+|\mathcal{G}_k) &= E(p_{k+1}(\ln X_k + Z_{n-k})|\mathcal{G}_k) = \int_{\mathbb{R}} p_{k+1}(\ln X_k + z)f_{k+1}(z)dz \\ &= q_k(\ln X_k) = p_k(\ln(X_k - \lambda_{n-k})), \end{aligned}$$

where the last two equalities follow from (3.2, 3.3). By induction therefore

$$E(X_n^+) = E(X_n^+|\mathcal{G}_0) = q_0(\ln X_0) = q_0(\ln \lambda_n),$$

which completes the proof of the first assertion.

2) Since $p_n \geq 0$ and $f_k \geq 0$ for all k, it is obvious that $p_k, q_k \geq 0$ for all k. We prove the upper bound by induction on k. The inequality $p_k(y) \leq a_k e^y + b_k$ holds for $k = n$ with $a_n = 1$ and $b_n = \lambda_0^+$. Assume $p_k(y) \leq a_k e^y + b_k$ holds for arbitrary $k \geq 1$. From (3.2)

$$\begin{aligned} q_{k-1}(x) &= \int_{\mathbb{R}} p_k(x+z)f_k(z)dz \leq \int_{\mathbb{R}} (a_k e^{x+z} + b_k)f_k(z)dz \\ &= a_k\mu_k e^x + b_k = a_{k-1}e^x + b_k, \end{aligned}$$

which means (3.5) holds. Now use (3.3) to obtain

$$\begin{aligned} p_{k-1}(y) &= q_{k-1}(\ln(e^y + \lambda_{n-k+1})) \leq a_{k-1}(e^y + \lambda_{n-k+1}) + b_k \\ &= a_{k-1}e^y + b_{k-1}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.2. 1) By induction on k we will prove

$$\begin{aligned} 0 &\leq p_k(x) - \tilde{p}_k(x) \leq \tilde{a}_k e^x + \tilde{b}_k \text{ for } x \in (L_k, U_k), 0 < k \leq n, \qquad \text{(B.1)} \\ 0 &\leq q_k(x) - \tilde{q}_k(x) \leq \tilde{a}_k (e^x - \lambda_{n-k}) + \tilde{b}_k \text{ for } x \in (\bar{L}_k, \bar{U}_k), 0 \leq k < n. \end{aligned}$$

Inequality (B.1) holds trivially for $k = n$. Suppose it holds for arbitrary $k < n$, then

$$q_{k-1}(x) - \tilde{q}_{k-1}(x) = \int_{\mathbb{R}} (p_k(x+z) - \tilde{p}_k(x+z)) f_k(z) dz \geq 0.$$

Since $(\bar{L}_{k-1}, \bar{U}_{k-1}) + (l_k, u_k) \subseteq (L_k, U_k)$ we have

$$\begin{aligned} q_{k-1}(x) - \tilde{q}_{k-1}(x) &= \int_{[l_k, u_k]} (p_k(x+z) - \tilde{p}_k(x+z)) f_k(z) dz \\ &\quad + \int_{\mathbb{R} \setminus [l_k, u_k]} (p_k(x+z) - \tilde{p}_k(x+z)) f_k(z) dz \\ &\leq \int_{[l_k, u_k]} (\tilde{a}_k e^{x+z} + \tilde{b}_k) f_k(z) dz + \int_{\mathbb{R} \setminus [l_k, u_k]} p_k(x+z) f_k(z) dz \\ &\leq \int_{\mathbb{R}} (\tilde{a}_k e^{x+z} + \tilde{b}_k) f_k(z) dz + \int_{\mathbb{R} \setminus [l_k, u_k]} (a_k e^{x+z} + b_k) f_k(z) dz \\ &= \tilde{a}_k \mu_k e^x + \tilde{b}_k + a_k e^x (F_k(l_k; 1) + G_k(u_k; 1)) \\ &\quad + b_k (F_k(l_k; 0) + G_k(u_k; 0)) \\ &= \tilde{a}_{k-1} e^x + \tilde{b}_{k-1} - \tilde{a}_{k-1} \lambda_{n-k+1} \end{aligned}$$

for $x \in (\bar{L}_{k-1}, \bar{U}_{k-1})$. Now $h_{k-1}((L_{k-1}, U_{k-1})) \subseteq (\bar{L}_{k-1}, \bar{U}_{k-1})$ which yields

$$\begin{aligned} 0 &\leq p_{k-1}(y) - \tilde{p}_{k-1}(y) = q_{k-1}(h_{k-1}(y)) - \tilde{q}_{k-1}(h_{k-1}(y)) \\ &\leq \tilde{a}_{k-1}(e^{h_{k-1}(y)} - \lambda_{n-k+1}) + \tilde{b}_{k-1} = \tilde{a}_{k-1} e^y + \tilde{b}_{k-1}. \end{aligned}$$

2, 3) The proof again proceeds by induction on k. Function $\tilde{p}_n$ is piecewise differentiable and satisfies (3.13). Assume (3.13) holds for arbitrary $k < n$. Function $0 \leq \tilde{p}'_k(x+z) f_k(z)$ is dominated by an integrable function for x in a compact interval therefore by Talvila (2001), Corollary 8 we can interchange integration and differentiation to obtain

$$0 \leq \tilde{q}'_{k-1}(x) = \int_{\mathbb{R}} \tilde{p}'_k(x+z) f_k(z) dz \leq a_{k-1} e^x + c_k \text{ for } x \in (\bar{L}_{k-1}, \bar{U}_{k-1}).$$

This in fact shows that $\tilde{q}'_{k-1}(x)$ is continuous in $(\bar{L}_{k-1}, \bar{U}_{k-1})$. Since $h'_{k-1}(y) \in (0, 1)$ for all y we also obtain

$$0 \leq \tilde{p}'_{k-1}(y) \leq \tilde{q}'_{k-1}(h_{k-1}(y)) \leq a_{k-1} e^{h_{k-1}(y)} + c_k = a_{k-1} e^y + c_{k-1},$$

which completes the proof.

4) Fix $k \leq n$. We have f_k, $\tilde{p}_k \in L^1(\mathbb{R})$ and let

$$g(x) = \int_{\mathbb{R}} \tilde{p}_k(x+z) f_k(z) dz = \tilde{p}_k * f_k(-z),$$

which by Young's inequality is also in $L^1(\mathbb{R})$. By the convolution theorem A.3,

$$\mathcal{F}(g) = \mathcal{F}(\tilde{p}_k * f_k(-z)) = \mathcal{F}(\tilde{p}_k)\mathcal{F}(f_k(-z)) = \mathcal{F}(\tilde{p}_k)\bar{\phi}_k.$$

By part 3) functions $\{\tilde{p}_k\}_{k=1}^{n-1}, \tilde{q}_k$ are continuously differentiable and bounded on the interior of their support therefore we conclude $\tilde{p}_k, \tilde{q}_k$ are also of finite variation on every compact interval. Since g coincides with $\tilde{q}_{k-1}$ on the interior of its support by Fourier inversion theorem A.2 we obtain (3.2). □

References

Andreasen, J. (1998). The pricing of discretely sampled Asian and lookback options: A change of numeraire approach. *Journal of Computational Finance 2*(1), 5–30.

Bayraktar, E. and H. Xing (2008). Pricing Asian options for jump diffusions. Available from `http://arxiv.org/abs/0707.2432`. To appear in Mathematical Finance.

Benhamou, E. (2002). Fast Fourier transform for discrete Asian options. *Journal of Computational Finance 6*(1), 49–68.

Bluestein, L. I. (1968). A linear filtering approach to the computation of the discrete Fourier transform. *IEEE Northeast Electronics Research and Engineering Meeting 10*, 218–219.

Carr, P., H. Geman, D. B. Madan, and M. Yor (2002). The fine structure of asset returns: An empirical investigation. *The Journal of Business 75*(2), 305–332.

Carverhill, A. and L. Clewlow (1990). Flexible convolution. *Risk 3*(4), 25–29.

Černý, A. (2004). Introduction to FFT in finance. *Journal of Derivatives 12*(1), 73–88.

Cont, R. and P. Tankov (2004). *Financial Modelling with Jump Processes*. Boca Raton: Chapman & Hall/CRC.

Fusai, G. and A. Meucci (2008). Pricing discretely monitored Asian options under Lévy processes. *Journal of Banking and Finance 32*(10), 2076–2088.

Goldberg, R. R. (1961). *Fourier transforms*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 52. New York: Cambridge University Press.

Ingersoll, J. E. (1987). *Theory of Financial Decision Making*. Studies in Financial Economics. Savage: Rowman & Littlefield.

Kemna, A. and T. Vorst (1990). A pricing method for options based on average asset values. *Journal of Banking and Finance 14*(1), 113–129.

Kushner, H. J. and P. Dupuis (2001). *Numerical Methods for Stochastic Control Problems in Continuous Time* (2nd ed.). New York: Springer.

Linetsky, V. (2004). Spectral expansions for Asian (average price) options. *Operations Research 52*(6), 856–867.

Lord, R., F. Fang, F. Bervoets, and C. W. Oosterlee (2008). A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes. *SIAM Journal on Scientific Computing 30*(4), 1678–1705.

Madan, D., P. Carr, and E. Chang (1998). The variance gamma process and option pricing. *European Finance Review 2*(1), 79–105.

Rabiner, L. R., R. W. Schafer, and C. M. Rader (1969). The chirp z-transform algorithm and its application. *Bell Systems Technical Journal 48*(5), 1249–1292.

Rogers, L. and Z. Shi (1995). The value of an Asian option. *Journal of Applied Probability 32*(4), 1077–1088.

Strawderman, R. L. (2004). Computing tail probabilities by numerical Fourier inversion: The absolutely continuous case. *Statistica Sinica 14*(1), 175–201.

Talvila, E. (2001). Necessary and sufficient conditions for differentiating under the integral sign. *The American Mathematical Monthly 108*(6), 544–548.

Turnbull, S. and L. Wakeman (1991). A quick algorithm for pricing European average options. *The Journal of Financial and Quantitative Analysis 26*(2), 377–389.

Večeř, J. (2001). New pricing of Asian options. *Journal of Computational Finance 4*(4), 105–113.

Večeř, J. (2002). Unified Asian pricing. *Risk 15*(6), 113–116.

Večeř, J. and M. Xu (2004). Pricing Asian options in a semimartingale model. *Quantitative Finance 4*(2), 170–175.

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