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# Quantile Autoregressive Distributed Lag Model with an Application to House Price Returns

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**Abstract**

This paper studies quantile regression in an autoregressive dynamic framework with exogenous stationary covariates. Hence, we develop a quantile autoregressive distributed lag model (QADL). We show that these estimators are consistent and asymptotically normal. Inference based on Wald and Kolmogorov-Smirnov tests for general linear restrictions is proposed. An extensive Monte Carlo simulation is conducted to evaluate the properties of the estimators. We demonstrate the potential of the QADL model with an application to house price returns in the United Kingdom. The results show that house price returns present a heterogeneous autoregressive behavior across the quantiles. The real GDP growth and interest rates also have an asymmetric impact on house prices variations.

Key Words: quantile autoregression, distributed lag model, autoregressive model

JEL Classification: C14; C32

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# 1 Introduction

Lags are present in econometrics for several reasons. Among these: price stickiness, psychological inertia (habit), permanent *versus* transitory shocks, adjustment costs, delays in implementing new technologies, etc. Modeling dynamic economic behavior has been a concern in econometrics, and constant-coefficient linear time series models have played an enormous role, see for instance Hendry and Richard (1982, 1983), Hendry, Pagan, and Sargan (1984) and Kiviet and Dufour (1997).

However, using constant-coefficient models as representations of economic time series may be insufficiently flexible. These models ignore the effects that a succession of small and varied macroeconomic shocks may have on the structure of dynamic economic models, particularly for highly aggregated data series. Moreover, these models cannot appropriately account for the presence of asymmetric dynamic responses. Of particular interest are the asymmetric business cycle dynamics over different quantiles of the economic variables. For instance, Beaudry and Koop (1993) show that positive shocks to the U.S. GDP are more persistent than negative shocks. Poterba (1991) and Capozza, Hendershott, Mack, and Mayer (2002) among others, present evidence on the asymmetric responses of house prices to income shocks. The occurrence of these asymmetries may therefore call into question the usefulness of models with time invariant structures as means of modeling such series. Quantile regression is a statistical method for estimating models of conditional quantile functions, which offers a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution, therefore exposing a variety of heterogeneity in response dynamics.

There is a growing literature about estimation of quantile regression models for time series. Koenker and Zhao (1996) extend quantile regression to linear ARCH models and studied a model with a pure location-shift. Engle and Manganelli (2004) propose a quantile autoregressive framework to model value-at-risk where the quantiles themselves follow an autoregressive process. Koenker and Xiao (2006a) consider quantile autoregression models in which the autoregressive coefficients can be expressed as monotone functions of a single, scalar random variable. Finally, Xiao (2006) studies quantile regression with cointegrated time series.

The purpose of this paper is to generalize the quantile autoregressive framework, proposed by Koenker and Xiao (2006a), by allowing for exogenous stationary covariates and to provide an application to illustrate the usefulness of the new model to study asymmetric behavior in time series. We develop a quantile autoregressive distributed lag (QADL) model. The QADL model can deliver important insights about asymmetric dynamics, such as heterogeneous adjustments in time series models where controlling for lagged regressors and exogenous covariates is important. The approach proposed in this paper is different from that of Engle and Manganelli (2004) because we use quantile regression in the standard linear time series context, modeling the conditional quantile function as linear and depending on past values of the dependent variable, instead of modeling the quantile functions themselves as an autoregressive process. This reduces the computational burden substantially. Moreover, the QADL model allows for some forms of explosive behavior in some quantiles while maintaining stationarity of the process, as long as certain stationarity conditions are satisfied on the whole distribution, while Engle and Manganelli (2004) exclude this case. It is important to note that we do not consider the Xiao (2006) case where the variables are cointegrated, but rather we consider an exogenous set of stationary covariates.

We illustrate the QADL model with an application to quarterly house price returns data in the United Kingdom (UK). House prices volatility has claimed unprecedented importance and there is a growing literature on this topic (for instance Muellbauer and Murphy, 1997;

Ortalo-Magné and Rady, 1999, 2006; Rosenthal, 2006). We argue that quantile regression can be used to describe the asymmetric responses of house prices returns to income and interest rates shocks. We interpret the conditional quantile functions as different phases of the house price cycle. The results show that house price returns have an asymmetric autoregressive behavior, and that real GDP growth and interest rates have an asymmetric impact on house prices returns along the quantiles. In addition, the results suggest that unit root behavior is present only in the high extreme quantiles. Thus, the model seems to show global stationarity with some persistence in unusually high returns. The inclusion of stationary covariates reduces the asymmetric autoregressive responses but maintains the persistence in the high quantiles. The interest rates have a negative impact on house prices returns, mostly significant for low quantiles. This can be interpreted as the fact that the interest rates have an effect on stimulating the demand in the real estate market when returns are low, but it does not deter house prices booms. In addition, there is evidence that the impact of GDP on house prices presents an asymmetric impact and it is stronger for low and high quantiles. For low quantiles, this is interpreted as the fact that GDP growth reactivates the real estate market when returns are low, while it might be contributing to house prices' busts (as that in the early 1990's where a recession was accompanied by a significant decline in house prices). Moreover, it contributes to sustaining house prices booms. In other words, periods of unusually high returns are very responsive to GDP growth. In fact, there is some evidence of overshooting for extreme high quantiles. In this case, the conditional mean may be a misleading estimator in periods of low and high returns, which are those when policymakers are more keen to intervene or to predict future behavior.

It is important to note that quantile autoregression in time series has a different interpretation than that of quantile regression in cross-sectional data. In general, quantile regression shows how a given quantile of the conditional distribution of $y$ depends on the covariates $x$.

In the cross-sectional case, this can be interpreted as the different effects that covariates exert on a given outcome for individuals on that corresponding quantile of the conditional distribution. In a time series context, however, we estimate the conditional quantile function of a particular variable along time, for instance aggregated variables such as GDP and consumption or index numbers. In the illustration presented in the paper we analyze price returns and we may interpret the conditional quantiles function at a given time as different phases of business cycles, where low and high quantiles of the conditional distribution of price returns corresponds to periods of declining and increasing prices respectively. This interpretation might also be used for output gap, consumption growth or value-at-risk applications.

The rest of the paper is organized as follows. Section 2 presents the model and assumptions. In Section 3 we describe the estimation and asymptotic properties of the estimators. In particular, we show that the QADL estimator is consistent and asymptotically normal. Section 4 develops the inference procedure and proposes a Wald type test for general linear hypotheses and a Kolmogorov-Smirnov test for linear hypothesis over a range of quantiles. Section 5 presents Monte Carlo evidence. In Section 6 we illustrate the new approach by applying it to the house price returns dataset. Finally, Section 7 concludes the paper.

# 2 The Model and Assumptions

The autoregressive-distributed lag model is described by the following equation

$$y_t = \mu + \sum_{j=1}^{p} \alpha_j y_{t-j} + \sum_{l=0}^{q} x'_{t-l}\theta_l + \varepsilon_t; \quad t = 1, ..., n \tag{1}$$

where $y_t$ is the response variable, $y_{t-j}$ is the lag of the response variable, $x_t$ is a $dim(x)$-dimensional vector of covariates and $\varepsilon_t$ is assumed to be white noise.[1] The main aim of this type of model is to emphasize alternative short-run dynamic structures. In addition, this

[1]We assume, for convenience, that each variable in $x_t$ have the same lag truncation, $q$. The case of different lag truncation for each variable is immediate.

class of models also provides important long-run results that are of particular interest for inference about the validity of a proposed economic theory. Nevertheless, the least squares based models might be insufficient to describe heterogeneity in the impact of the shocks in a given time series. Quantile autoregressive distributed lag model may be viewed as a model which complements the model described in equation (1).

As in Koenker and Xiao (2006a), let $\{U_t\}$ be a sequence of $i.i.d.$ standard uniform random variables, and consider the following autoregressive-distributed lag process

$$y_t = \mu\left(U_t\right) + \sum_{j=1}^{p} \alpha_j\left(U_t\right) y_{t-j} + \sum_{l=0}^{q} x'_{t-l}\theta_l\left(U_t\right) \tag{2}$$

where $\alpha$ and $\theta$ are unknown functions [0,1]$\rightarrow \mathbb{R}$ that we want to estimate. Given that the right hand side of (2) is monotone increasing on $U_t$, it follows that the $\tau$-th conditional quantile function of $y_t$ can be written as

$$Q_{y_t}\left(\tau|\Im_t\right) = \mu\left(\tau\right) + \sum_{j=1}^{p} \alpha_j\left(\tau\right) y_{t-j} + \sum_{l=0}^{q} x'_{t-q}\theta_l\left(\tau\right) \tag{3}$$

where $\Im_t$ is the $\sigma$-field generated by $\{y_s, x_s, s \leq t\}$.[2] Implicitly in the formulation of model (3) is the requirement that $Q_{y_t}(\tau|\Im_t)$ is monotone increasing in $\tau$ for all $\Im_t$. A more compact notation to describe model (3) is

$$Q_{y_t}\left(\tau|\Im_t\right) = z'_t\beta(\tau)$$

where $z_t = (1, y_{t-1}, ..., y_{t-p}, x_t, ..., x_{t-q})'$ and $\beta(\tau) = (\mu\left(\tau\right), \alpha_1\left(\tau\right), ..., \alpha_p\left(\tau\right), \theta'_0\left(\tau\right), ..., \theta'_q\left(\tau\right))'$.

It is important to emphasize that monotonicity of the conditional quantile functions imposes some discipline on the forms taken by the coefficients. This essentially requires that

[2]The transition from (2) to (3) is an immediate consequence of the fact that for any monotone increasing function g and standard uniform random variable, $U$, we have

$$Q_{g(U)}(\tau) = g(Q_U(\tau)) = g(\tau),$$

where $Q_U(\tau) = \tau$ is the quantile function of $U$.

the function $Q_{y_t}(\tau|\Im_t)$ is monotone in $\tau$ in some relevant region of $\Im_t$-space. In some circumstances, this necessitates restricting the domain of the dependent variables; in other cases, when the coordinates of the dependent variables are themselves functionally dependent, monotonicity may hold globally. The estimated conditional quantile function

$$\hat{Q}_{y_t}(\tau|\Im_t) = z_t'\hat{\beta}(\tau) \tag{4}$$

is ensured to be monotone in $\tau$ at $z_t = \bar{z}$, as noted in Koenker and Xiao (2006a). However, monotonicity at $z_t = \bar{z}$ does not guarantee that $\hat{Q}_{y_t}(\tau|\Im_t)$ will be monotone in $\tau$ for other values of $z$. Furthermore, once we are using a liner model, there must be crossing sufficiently far away from $\bar{z}$.[3] It may be that such crossing occurs outside the convex hull of the $z$ observations, in which case the estimated model may be viewed as an adequate approximation within this region. But it is not unusual to find that the crossing has occurred in this region as well. As discussed in Koenker and Xiao (2006a), one can find a linear reparametrization of the model that does exhibit comonotonicity over some relevant region of covariate space.

It is easy to check whether $\hat{Q}_{y_t}(\tau|\Im_t)$ is monotone at particular $z$ points. In order to verify monotonicity for a given $z$, one may compute equation (4) for several quantiles, evaluated at such $z$, and plot it against the sequence of $\tau$'s. If there is a significant number of observed points at which this condition is violated, then this can be taken as evidence of model misspecification. Failure of the monotonicity condition might also imply that the conditional quantile functions are not linear. In this paper, we assume that monotonicity of $Q_{y_t}(\tau|\Im_t)$ in $\tau$, for some relevant region of $\Im_t$-space, holds.[4]

The quantile autoregressive distributed-lag of orders $p$ and $q$ (QADL$(p, q)$) model (3) can

[3]Neocleous and Portnoy (2008) show that if one considers grids for $\tau \in [0, 1]$ with spacing $\delta_n$ wider than $O(1/(n \log(n)))$ for the full quantile regression process, and satisfying $\limsup \delta_n n^{\eta} > 0$ and $\liminf \delta_n n^{1/2}/\log n > 0$ for some $\eta > 0$, then with probability tending to one, $\hat{Q}_{y_t}(\tau|\Im_t)$ is strictly monotone for $\epsilon \leq \tau \leq 1 - \epsilon$ and bounded domain for $z$.

[4]We refer the reader to Koenker and Xiao (2006a), Koenker and Xiao (2006b), and Koenker (2005) for more details about monotonicity in quantile regression.

be written in more conventional random-coefficient notation as

$$y_t = \mu_0 + \sum_{j=1}^{p} \alpha_{j,t} y_{t-j} + \sum_{l=0}^{q} x'_{t-l} \theta_{l,t} + u_t, \tag{5}$$

where $\mu_0 = E\mu(U_t)$, $u_t = \mu(U_t) - \mu_0$, $\alpha_{j,t} = \alpha_j(U_t)$ and $\theta_{l,t} = \theta_l(U_t)$ for $j = 1, 2, \cdots, p$ and $l = 0, 1, \cdots, q$. Thus $\{U_t\}$ is an independent and identically distributed ($i.i.d.$) sequence of random variables with distribution function $F(\cdot) = \mu^{-1}(\cdot + \mu_0)$ and the $\alpha_{j,t}$ and $\theta_{l,t}$ coefficients are functions of the $u_t$ innovation random variable.

In model (5) the choice of $p$ and $q$ are important. In order to select appropriated models we suggest the use of BIC criteria, adapted to QADL along the lines suggested by Machado (1993), which is based on the Asymmetric Laplace Distribution. At the median it uses the criterion

$$BIC = n \log \hat{\sigma} + \frac{1 + p + (1 + q) \times dim(x)}{2} \log n$$

where $\hat{\sigma} = n^{-1} \sum |y_t - z'_t \hat{\beta}(1/2)|$. For other quantiles, the obvious asymmetric modification of this expression can be used. In the example given in this paper we select the number of lags based only on the median criterion, in order to have a comparable regression model across quantiles. But, it is possible that there are applications in which this is not desirable.

For stationarity and asymptotic analysis, we introduce the following assumptions.

**A.1:** $\{u_t\}$ are $iid$ random variables with mean 0 and variance $\sigma^2 < \infty$.

**A.2:** The distribution function of $u_t$, $F$, has a continuous Lebesgue density, $f$, with $0 < f(u) < \infty$ on $\mathcal{U} = \{u : 0 < F(u) < 1\}$.

**A.3:** Let $E(A_t \otimes A_t) = \Omega_A$ where $A_t = \begin{bmatrix} A_{p-1,t} & \alpha_{p,t} \\ \mathbf{I}_{p-1} & \mathbf{0}_{(p-1)\times 1} \end{bmatrix}$ and $A_{p-1,t} = [\alpha_{1,t}, \alpha_{2,t}, \cdots, \alpha_{p-1,t}]$. Then, the eigenvalues of $\Omega_A$ have moduli less than unity.

**A.4:** Let $E(\Theta_t \otimes \Theta_t) = \Omega_\Theta$, where $\Theta_t = \begin{bmatrix} \Theta_{q-1,t} & \theta_{q,t} \\ \mathbf{0}_{(p-1)\times(q-1)} & \mathbf{0}_{(p-1)\times 1} \end{bmatrix}$ and $\Theta_{q-1,t} = [\theta_{0,t}, \theta_{1,t}, \cdots, \theta_{q-1,t}]$. $\{x_t\}$ is a weakly stationary sequence, and $\Omega_\Theta$ exists;

**A.5:** $\tau \in \mathcal{T} = [c, 1-c]$ with $c \in (0, 1/2)$. For all $\tau$, $(\alpha(\tau), \theta'(\tau)) \in \text{int } \mathscr{A} \times \mathscr{G}$, and $\mathscr{A} \times \mathscr{G}$ is compact and convex; and $\max_t \|x_t\| = O(\sqrt{T})$; $\max_t \|y_t\| = O(\sqrt{T})$;

**A.6:** Denote the conditional distribution function $\Pr[y_t < \cdot | \Im_t]$ as $F_{t-1}(\cdot)$ and its derivative as $f_{t-1}(\cdot)$, $f_{t-1}$ is uniformly integrable on $\mathcal{U}$.

A.1, A.2, A.5 and A.6 are standard assumptions in the quantile regression framework. A.3 deserves some scrutiny. It restricts the non-stationary behavior of the dependent variable. However, as shown in Theorem 1, the process may allow for some forms of explosive behavior in some quantiles while maintaining stationarity in the quantile process. Assumption A.4 restricts the exogenous covariates to be stationary.

Under some regularity conditions, the next theorem derives the stochastic behavior of $y_t$ and it will facilitate the asymptotic analysis in the next section.

**Theorem 1** *Under assumptions A.1 - A.4, the time series $y_t$ given by (5) is covariance stationary.*

# 3 Estimation of QADL

In this section we describe the estimation method and study the asymptotic properties of these estimators. The estimation procedure is based on standard linear quantile regression. Thus, estimation of the quantile autoregressive distributed lag model (3) involves solving the following problem

$$\min_{\beta \in \Re^{1+p+(1+q)\times dim(x)}} \sum_{t=1}^{n} \rho_\tau \left(y_t - z_t'\beta\right) \tag{6}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$, as in Koenker and Bassett (1978). We are mostly concerned with the asymptotic properties of the $\hat{\beta}(\tau)$ coefficients in (6). First we state consistency of the estimators and later asymptotic normality. The proofs of the theorems appear in the Appendix.

Under the assumptions discussed above, the estimator $\hat{\beta}(\tau)$ is consistent. The following theorem formalizes the result.

**Theorem 2** *Under Assumptions A1-A6,*

$$\hat{\beta}(\tau) \xrightarrow{p} \beta(\tau).$$

The consistency of the estimators is achieved by the argmax theorem in van der Vaart and Wellner (1996).

Given the estimates, $\hat{\beta}(\tau)$, the $\tau-$th conditional quantile function of $y_t$, conditional of $z_t$, can be estimated by

$$\hat{Q}_{y_t}(\tau|z_t) = z_t'\hat{\beta}(\tau).$$

In addition, given a family of estimated conditional quantile functions, the conditional density of $y_t$ at various values of the conditioning covariate can be estimated by the difference quotients,

$$\hat{f}_{y_t}(\tau|z_t) = (\tau_i - \tau_{i-1})/\left(\hat{Q}_{y_t}(\tau_i|z_t) - \hat{Q}_{y_t}(\tau_{i-1}|z_t)\right),$$

for some appropriately chosen sequence of $\tau'$s.

Now we move our attention to the asymptotic normality of the estimators. In order to derive the limiting distribution of the estimators define $\hat{v} = n^{1/2}(\hat{\beta}(\tau) - \beta(\tau))$, and write $\rho_\tau(y_t - \hat{\beta}(\tau)'z_t)$ as $\rho_\tau(u_{t\tau} - (n^{-1/2}\hat{v})'z_t)$ where $u_{t\tau} = y_t - z_t'\beta(\tau)$. Minimization of (6) is equivalent to the following problem:

$$\min_v \sum_{t=1}^{n}[\rho_\tau(u_{t\tau} - (n^{-1/2}v)'z_t) - \rho_\tau(u_{t\tau})]. \tag{7}$$

Note that $\hat{v}$ is a minimizer of $H_n(v) = \sum_{t=1}^{n}\left[\rho_\tau(u_{t\tau} - (n^{-1/2}v)'z_t) - \rho_\tau(u_{t\tau})\right]$. The objective function $H_n(v)$ is a convex random function. Knight (1989, 1991) and Pollard (1991) show that if the finite-dimensional distributions of $H_n(\cdot)$ converge weakly to those of $H(\cdot)$ and $H(\cdot)$ has a unique minimum, the convexity of $H_n(\cdot)$ implies that $\hat{v}$ converges in distribution to the minimizer of $H(\cdot)$. Denoting $\psi_\tau(u) = \tau - I(u < 0)$ for $u \neq 0$, following the approach of Knight (1989) it can be shown that the limiting distribution of $\hat{v}$ is determined by the limiting behavior of the function $H_n(v)$. Using Knight's identity,

$$\begin{aligned}\rho_\tau(u-v) - \rho_\tau(u) &= -v\psi_\tau(u) + (u-v)\{I(0 > u > v) - I(0 < u < v)\} \\ &= -v\psi_\tau(u) + \int_0^v \{I(u \leq s) - I(u < 0)\} ds\end{aligned}$$

the objective function for minimization of problem (7) can be rewritten as

$$\begin{aligned}H_n(v) &= \sum_{t=1}^{n} [\rho_\tau(u_{t\tau} - (n^{-1/2}v)'z_t) - \rho_\tau(u_{t\tau})] \\ &= -\sum_{t=1}^{n} v'n^{-1/2} z_t \psi_\tau(u_{t\tau}) + \sum_{t=1}^{n} \int_0^{(n^{-1/2}v)'z_t} \{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\} ds.\end{aligned}$$

Therefore, in order to derive the asymptotic results for the limiting distribution of $n^{1/2}(\hat{\beta}(\tau) - \beta(\tau))$ we need to study the convergence of the two terms of $H_n(v)$ in the above equation, $n^{-1/2}\sum_{t=1}^{n} v'z_t\psi_\tau(u_{t\tau})$ and $\sum_{t=1}^{n}\int_0^{(n^{-1/2}v)'z_t}\{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\}ds$. Thus, once we show that $H_n(\cdot)$ converge weakly to $H(\cdot)$ we just need to find the minimizer of $H(\cdot)$, and $\hat{v}$ converges in distribution to that minimizer.

Define the following elements: $E(y_t) = \mu_y$; $E(x_t) = \mu_x$; $E(y_t y_{t-j}) = \gamma_{y_j}$; $E(x_t x_{t-j}) = \gamma_{x_j}$; $E(y_t x_{t-j}) = \gamma_{yx_j}$; $\Omega_0 = E(z_t z_t') = \lim n^{-1}\sum_{t=1}^{n} z_t z_t'$ then

$$\Omega_0 = \begin{bmatrix} 1 & \mu_y & \mu_x \\ \mu_y & \Omega_y & \Omega_{yx} \\ \mu_x & \Omega_{yx} & \Omega_x \end{bmatrix},$$

$$\Omega_y = \begin{bmatrix} \gamma_{y_0} & \cdots & \gamma_{y_{p-1}} \\ \vdots & \ddots & \vdots \\ \gamma_{y_{p-1}} & \cdots & \gamma_{y_0} \end{bmatrix}, \ \Omega_x = \begin{bmatrix} \gamma_{x_0} & \cdots & \gamma_{x_q} \\ \vdots & \ddots & \vdots \\ \gamma_{x_q} & \cdots & \gamma_{x_0} \end{bmatrix}, \ \Omega_{yx} = \begin{bmatrix} \gamma_{yx_0} & \cdots & \gamma_{yx_{p+q}} \\ \vdots & \ddots & \vdots \\ \gamma_{yx_{p+q}} & \cdots & \gamma_{yx_0} \end{bmatrix}.$$

In addition, let $\Omega_1(\tau) = \lim n^{-1}\sum_{t=1}^{n} f_{t-1}[F_{t-1}^{-1}(\tau)]z_t z_t'$ and $\Sigma(\tau) = \Omega_1(\tau)^{-1}\Omega_0\Omega_1(\tau)^{-1}$. We will discuss estimation of $\Sigma(\tau)$ in the next section. The limiting distribution of the quantile autoregressive distributed lag estimator is given in Theorem 3. The proof of the theorem is given in the Appendix.

**Theorem 3** *Under Assumptions A1-A6,*

$$\Sigma(\tau)^{-1/2}\sqrt{n}\left(\hat{\beta}(\tau) - \beta(\tau)\right) \Rightarrow B_k(\tau)$$

*where $B_k(\tau)$ represents a k-dimensional Brownian Bridge where $k = 1+p+(1+q)\times dim(x)$.*

By definition, for any fixed $\tau$, $B_k(\tau)$ is $N(0, \tau(1-\tau)I_k)$. Therefore, for the important case of estimation of a fixed quantile $\tau$, we state the result in the following corollary.

**Corollary 1** *Under Assumptions A1-A6, for a fixed $\tau$,*

$$\sqrt{n}\left(\hat{\beta}(\tau) - \beta(\tau)\right) \Rightarrow N(0, \tau(1-\tau)\Sigma(\tau)),$$

**Remark 1** *In the special case of fixed quantile $\tau$, and constant coefficients, $\Omega_1(\tau) = f[F^{-1}(\tau)]\Omega_0$, where $f(\cdot)$ and $F(\cdot)$ are the density and distribution functions of $u_t$, respectively. Thus, Corollary 1 simplifies to*

$$f[F^{-1}(\tau)]\Omega_0^{-1/2}\sqrt{n}\left(\hat{\beta}(\tau) - \beta(\tau)\right) \Rightarrow N(0, \tau(1-\tau)I_k).$$

# 4 Inference on QADL

In this section, we turn our attention to inference in the quantile autoregression distributed lag model (QADL), and suggest a Wald type test for general linear hypotheses, and a Kolmogorov-Smirnov test for linear hypothesis over a range of quantiles $\tau \in \mathcal{T}$. In the independent and identically distributed setup the conditional quantile functions of the response variable, given the covariates, are all parallel, implying that covariates effects shift the location of the response distribution but do not change the scale or shape. However, slopes estimates often vary across quantiles implying that it is important to test for equality of slopes across quantiles. Wald tests designed for this purpose were suggested by Koenker and Bassett (1982a), Koenker and Bassett (1982b), and Koenker and Machado (1999). It is possible to formulate a wide variety of tests using variants of the proposed Wald test, from simple tests on a single quantile regression coefficient to joint tests involving many covariates and distinct quantiles at the same time.

General hypotheses on the vector $\beta(\tau)$ can be accommodated by Wald-type tests. The Wald process and associated limiting theory provide a natural foundation for the hypothesis $R\beta(\tau) = r$, when $r$ is known. Here $R$ is a $k \times (1 + p + (1 + q)dim(x))$ matrix with rank $k$ and $r$ is a $k$-dimensional vector.

Under the null hypothesis $H_0 : R\beta(\tau) = r$, conditions A1-A6, we have

$$\mathcal{V}_n(\tau) = \sqrt{n}[R\Sigma(\tau)R']^{-1/2}(R\hat{\beta}(\tau) - r) \Rightarrow B_k(\tau), \tag{8}$$

where $B_k(\tau)$ represents a $k$-dimensional standard Brownian Bridge. For any fixed $\tau$, $B_k(\tau)$ is $N(0, \tau(1 - \tau)I_k)$. The normalized Euclidean norm of $B_k(\tau)$

$$Q_k(\tau) = \|B_k(\tau)\|/\sqrt{\tau(1 - \tau)}$$

is generally referred to as a Kiefer process of order $k$. Moreover, for given $\tau$, the regression Wald process can be constructed as

$$\mathcal{W}_n(\tau) = \frac{n(R\hat{\beta}(\tau) - r)'[R\hat{\Omega}_1(\tau)^{-1}\hat{\Omega}_0\hat{\Omega}_1(\tau)^{-1}R']^{-1}(R\hat{\beta}(\tau) - r)}{\tau(1 - \tau)} \tag{9}$$

where $\hat{\Omega}_1(\tau)$ and $\hat{\Omega}_0$ are consistent estimators of $\Omega_1(\tau)$ and $\Omega_0$, respectively.

Under $H_0$, the statistic $\mathcal{W}_n$ is asymptotically $\chi^2_k$ with $k$-degrees of freedom. The limiting distribution of the test is summarized in the following theorem:

**Theorem 4** *(Wald Test Inference). Under $H_0 : R\beta(\tau) = r$, and conditions A1-A6, for fixed $\tau$*

$$\mathcal{W}_n(\tau) \Rightarrow \chi^2_k.$$

**Proof.** The proof is simple and it follows from observing that for any fixed $\tau$, by Corollary 1

$$\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \Rightarrow N(0, \tau(1-\tau)\Sigma(\tau))$$

under the null hypothesis,

$$\sqrt{n}(R\hat{\beta}(\tau) - r) \Rightarrow N(0, \tau(1-\tau)R\Sigma(\tau)R')$$

since $\hat{\Sigma}(\tau)$ is a consistent estimator of $\Sigma(\tau)$, by Slutsky theorem

$$\mathcal{W}_n(\tau) = \frac{n(R\hat{\beta}(\tau) - r)'[R\hat{\Sigma}(\tau)R']^{-1}(R\hat{\beta}(\tau) - r)}{\tau(1-\tau)} \Rightarrow \chi^2_k.$$

■

In order to implement the test it is necessary to estimate $\Sigma(\tau)$ consistently. There are several consistent estimators available. Since $\Omega_0$ involves no nuisance parameter it can easily be estimated as

$$\hat{\Omega}_0(\tau) = \frac{1}{n}\sum_{t=1}^{n} z_t z_t'.$$

Two approaches to the estimation of the matrix $\Omega_1(\tau)$ will be described. First, we describe the approach proposed by Bassett and Koenker (1982) and Hendricks and Koenker (1991). Provided that the $\tau$ conditional quantile function of $y|z$ is linear, then for $h_n \to 0$ we can consistently estimate the parameters of the $\tau \pm h_n$ conditional quantile function by $\hat{\beta}(\tau \pm h_n)$, and the density $f_t(\xi_t)$ can thus be estimated by the difference quotient

$$\hat{f}_t(\xi_t) = \frac{2h_n}{z_t'(\hat{\beta}(\tau + h_n) - \hat{\beta}(\tau - h_n))} \tag{10}$$

where $h_n$ is a bandwidth which tends to zero as $n \to \infty$, and $\xi_t = z_t'\beta(\tau)$. Substituting this estimator in the expression for $\Omega_1(\tau)$ yields the feasible estimator for the asymptotic covariance matrix. Following Powell (1986), the second approach of the estimator of $\Omega_1(\tau)$ takes the form

$$\hat{\Omega}_1(\tau) = \frac{1}{2nh_n} \sum_{t=1}^{n} I(|\hat{u}_t(\tau)| \leq h_n) z_t z_t',$$

where $\hat{u}_t(\tau) = y_t - z_t'\hat{\beta}(\tau)$ and $h_n$ is an appropriately chosen bandwidth, with $h_n \to 0$ and $nh_n^2 \to \infty$. The consistency of these estimators is standard and will not be discussed in this paper. For the Monte Carlo experiments and the application we only consider the second method.

More general hypotheses are also easily accommodated by the Wald approach. Let $\upsilon = (\beta(\tau_1)', ..., \beta(\tau_m)')$ and define the null hypothesis as $H_0 : R\upsilon = r$, where $R$ is a $k \times (m \times (1 + p + (1 + q)dim(x)))$ matrix of rank $k$ and $r$ is a $k$-dimensional vector. The test statistic is similar to the Wald test in equation (9),

$$\mathcal{W}_n = n(R\hat{\upsilon} - r)'[RVR']^{-1}(R\hat{\upsilon} - r).$$

However, the asymptotic covariance matrix for $\beta(\tau_1), ..., \beta(\tau_m)$ has blocks

$$V(\tau_i, \tau_j) = [\tau_i \wedge \tau_j - \tau_i\tau_j]\Omega_1(\tau_i)^{-1}\Omega_0\Omega_1(\tau_j)^{-1},$$

where $\Omega_1(\tau)^{-1}$ and $\Omega_0$ are estimated as above. The statistic $\mathcal{W}_n$ is still asymptotically $\chi_k^2$ under $H_0$. This formulation accommodates a wide variety of testing situations, from a simple test on single quantiles regression coefficients to joint tests involving several parameters and distinct quantiles. Thus, for instance, we might test for the equality of several slope coefficients across several quantiles.

Another important class of tests in the quantile regression literature involves the Kolmogorov-Smirnov (KS) type tests, where the interest is to examine the property of the estimator over a range of quantiles $\tau \in \mathcal{T}$, instead of focusing only on selected quantiles. Thus, if one has interest in testing $R\beta(\tau) = r$ over $\tau \in \mathcal{T}$, one may consider the KS type sup-Wald test. Following Koenker and Xiao (2006a), we may construct a KS type test on the quantile autoregression distributed lag process in the following way:

$$KS\mathcal{W}_n = \sup_{\tau \in \mathcal{T}} \mathcal{W}_n(\tau). \tag{11}$$

The limiting distribution of the Kolmogorov-Smirnov test is given in the following theorem:

**Theorem 5** (Kolmogorov-Smirnov Test). Under $H_0$ and conditions A1-A6,

$$KS\mathcal{W}_n = \sup_{\tau \in \mathcal{T}} \mathcal{W}_n(\tau) \Rightarrow \sup_{\tau \in \mathcal{T}} Q_k^2(\tau).$$

The proof of Theorem 5 follows directly from the continuous mapping theorem and equation (8). Critical values for $\sup Q_k^2(\tau)$ have been tabled by DeLong (1981) and, more extensively, by Andrews (1993) using simulation methods.

# 5 Monte Carlo

## 5.1 Monte Carlo Design

In this section, we describe the design of some simulation experiments to assess the finite sample performance of the QADL estimator and the inference procedures discussed in the previous section. Two simple versions of the basic model (3) are considered in the simulation experiment. In the first version, reported in Tables 1 and 2, the scalar covariate, $x_t$, exerts a pure location shift effect. In the second, reported in Tables 3 and 4, $x_t$ exerts both location and scale effects. In the former case the response $y_t$ is generated by the model,

$$y_t = \mu + \alpha y_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + u_t, \tag{12}$$

while in the latter,

$$y_t = \mu + \alpha y_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + (\gamma x_t) u_t. \tag{13}$$

We employ three different schemes to generate the disturbances $u_t$. Under scheme 1, we generate $u_t$ as $N(0, \sigma_u^2)$. Under scheme 2 we generate $u_t$ as $t$-distribution with 3 degrees of freedom.

In all cases we set $y_0 = 0$ and generate $y_t$ for $t = 1, ..., n$ according to equations (12) and (13), and in generating $y_t$ we discarded the first 100 observations, using the remaining observations for estimation. This ensures that the results are not unduly influenced by the initial values of the $y_0$ process. We generate the exogenous covariates, $x_t$, using the same distribution as the innovations $u_t$, that is, we draw $x_t$ from a normal distribution under scheme 1, and from $t$-distribution with 3 degrees of freedom under scheme 2. In the simulations, we experiment with sample sizes $n = 100, 200$. We set the number of replications to 5000, and consider the following values for the remaining parameters:

$$(\alpha, \beta_1, \beta_2) = (0.5, 0.5, 0.5), (0.75, 0.75, 0.75);$$

$$\gamma = 0.5, \quad \sigma_u^2 = 1.$$

In the Monte Carlo study, we compare the estimators' coefficients in terms of bias and root mean squared error. We also investigate the small sample properties of the tests based on different estimators paying particular attention to size and power.

## 5.2 Monte Carlo Results

We study four different estimators in the Monte Carlo experiment, the quantile autoregression (QAR) proposed by Koenker and Xiao (2006a), the quantile autoregressive distributed lag model (QADL) proposed in this paper, the least squares estimator (OLS), and finally, the least square distributed lag model (ADL). The quantile regression based estimators are analyzed for the median case. We also consider different sample sizes in the experiments, however, because of the space limitations we report results only for $n = 100$. The results are similar for the $n = 200$ sample size design.

### 5.2.1 Bias and RMSE

In the first part of the Monte Carlo we study the bias and root mean squared error (RMSE) of the estimators. Tables 1 and 2 present bias and RMSE results of the estimators $\alpha$, $\beta_1$, and $\beta_2$ for location-shift model for all estimators. We present the simulation results for $(\alpha, \beta_1, \beta_2) = (0.5, 0.5, 0.5)$ only (but similar results are obtained for the other set of parameter values). For QAR and OLS models we do not include the terms $x_t$ and $x_{t-1}$ in the estimation equation. The results show that, as expected, omitting the variables in QAR and OLS cases cause bias in estimation. In addition, in general, the bias of the estimators is bigger for smaller coefficient values.

Table 1 shows the results for the first set of parameters with all three distributions. When the disturbances are sampled from a Gaussian distribution, as expected, the autoregression coefficient is biased for the QAR and OLS cases, and the QADL and ADL are approximately unbiased. Regarding the RMSE, in the Gaussian condition, the OLS based estimators for perform better than quantile regression estimators, that is, ADL has a smaller RMSE's when compared with QADL, and OLS has smaller RMSE when compared with QAR. However, for the non-Gaussian cases $t_3$ and chi-squared, the quantile regression base estimators perform better in terms of RMSE $vis-a-vis$ the least squared based estimators.

Table 2 presents the results for the second parameter case, where we increase the value of parameters. The results are essentially the same as those in Table 1. The autoregressive estimates of QAR and OLS are biased because of the omitted variables problem, evidencing that if the model contains an exogenous variable, omitting such variable will bias the results.

| | | QAR | QADL | OLS | ADL |
|---|---|---|---|---|---|
| Normal | $\alpha$ | $-0.0971$ | $-0.0087$ | 0.0939 | $-0.0081$ |
| | | (0.1311) | (0.0775) | (0.1116) | (0.0505) |
| | $\beta_1$ | – | 0.0014 | – | 0.0009 |
| | | – | (0.1109) | – | (0.0717) |
| | $\beta_2$ | – | 0.0037 | – | $-0.0034$ |
| | | – | (0.1172) | – | (0.0758) |
| $t_3$ | $\alpha$ | 0.0968 | $-0.0027$ | 0.0989 | $-0.0074$ |
| | | (0.0844) | (0.0389) | (0.1137) | (0.0497) |
| | $\beta_1$ | – | $-0.0005$ | – | 0.0006 |
| | | – | (0.0625) | – | (0.1215) |
| | $\beta_2$ | – | 0.0018 | – | 0.0071 |
| | | – | (0.0658) | – | (0.1239) |
| $\chi_3$ | $\alpha$ | 0.0715 | $-0.0004$ | 0.0976 | $-0.0079$ |
| | | (0.0979) | (0.0502) | (0.1129) | (0.0503) |
| | $\beta_1$ | – | 0.0053 | – | 0.0023 |
| | | – | (0.0728) | – | (0.0743) |
| | $\beta_2$ | – | 0.0045 | – | 0.0056 |
| | | – | (0.0767) | – | (0.0780) |

Table 1: Location-Shift Model: Bias and RMSE of Estimators ($n = 100$)

The results concerning RMSE are qualitatively the same and the QADL model performs well for a heavy tail non-Gaussian case.

Tables 3 and 4 present bias and RMSE results of the estimators $\alpha$ and $\beta$ for location-scale-shift model. As in the location case, the bias of the estimators decreases as the coefficients become larger. Again, the QADL estimator performs very well in the non-Gaussian heavy tail cases.

Table 3 shows that in the Gaussian case the QAR and OLS are biased and the QADL and ADL are approximately unbiased. As in the location case, in the presence of dynamic variables, the quantile autoregression estimator proposed by Koenker and Xiao (2006a) is biased if the true model indeed has exogenous covariates and one omits them when estimating the model. The RMSE's are smaller when compared with the previous location case, but present the same features. Table 4 presents the results for the $t_3$-distribution case, and the

| | | QAR | QADL | OLS | ADL |
|---|---|---|---|---|---|
| Normal | $\alpha$ | 0.0710 | $-0.0005$ | 0.0705 | $-0.0060$ |
| | | (0.0945) | (0.0457) | (0.0813) | (0.0301) |
| | $\beta_1$ | – | $-0.0010$ | – | $-0.0012$ |
| | | – | (0.1107) | – | (0.0737) |
| | $\beta_2$ | – | 0.0055 | – | 0.0042 |
| | | – | (0.1253) | – | (0.0821) |
| $t_3$ | $\alpha$ | 0.0372 | $-0.0018$ | 0.0789 | $-0.0062$ |
| | | (0.0565) | (0.0229) | (0.0801) | (0.0380) |
| | $\beta_1$ | – | 0.0007 | – | 0.0036 |
| | | – | (0.0626) | – | (0.1584) |
| | $\beta_2$ | – | 0.0015 | – | 0.0066 |
| | | – | (0.0640) | – | (0.1595) |
| $\chi_3$ | $\alpha$ | 0.0699 | $-0.0045$ | 0.0706 | $-0.067$ |
| | | (0.0910) | (0.0363) | (0.0809) | (0.0303) |
| | $\beta_1$ | – | 0.0003 | – | 0.0039 |
| | | – | (0.0722) | – | (0.0894) |
| | $\beta_2$ | – | 0.0040 | – | 0.0066 |
| | | – | (0.0752) | – | (0.0931) |

Table 2: Location-Shift Model: Bias and RMSE of Estimators ($n = 100$)

results are qualitatively similar those in Table 2.

### 5.2.2 Size and Power

Now we turn our attention to the size and power of the asymptotic inference given in the previous section. First, we concentrate on tests for selected quantiles, latter we move to tests over a range of quantiles. For the former case, in order to calculate the power curves we use the same setup as in the presented calculation of bias and RMSE. We present the results for QADL as well as for ADL in order to compare the finite sample performance of the estimators. Thus, we consider the model in equation (12) and test the hypothesis that $\hat{\alpha}(\tau) = \alpha$ and also that $\hat{\beta}(\tau) = \beta$ for given $\tau$. We present the results for $\alpha = 0.5$ and $\beta = 0.5$.[5] For models under the alternative, we considered linear deviations from the null as $\alpha + d/\sqrt{n}$ and $\beta + d/\sqrt{n}$. The construction of the test uses the density estimator given in

[5]The results for $\alpha = \beta_1 = \beta_2 = 0.75$ are similar and we omit them to save space.

| | | QAR | QADL | OLS | ADL |
|---|---|---|---|---|---|
| Normal | $\alpha$ | 0.1940 | $-0.0014$ | 0.1768 | $-0.0042$ |
| | | (0.2067) | (0.0341) | (0.1844) | (0.0336) |
| | $\beta_1$ | – | $-0.0009$ | – | $-0.0009$ |
| | | – | (0.0625) | – | (0.0355) |
| | $\beta_2$ | – | 0.0011 | – | 0.0028 |
| | | – | (0.0386) | – | (0.0373) |
| $t_3$ | $\alpha$ | 0.0764 | $-0.0006$ | 0.0977 | $-0.0052$ |
| | | (0.1015) | (0.0188) | (0.1404) | (0.1196) |
| | $\beta_1$ | – | 0.0027 | – | 0.0004 |
| | | – | (0.1456) | – | (0.1463) |
| | $\beta_2$ | – | 0.0002 | – | 0.0003 |
| | | – | (0.0286) | – | (0.1558) |
| $\chi_3$ | $\alpha$ | 0.0689 | $-0.0010$ | 0.0950 | $-0.0084$ |
| | | (0.0969) | (0.0495) | (0.1108) | (0.0505) |
| | $\beta_1$ | – | 0.0041 | – | 0.0002 |
| | | – | (0.0726) | – | (0.0741) |
| | $\beta_2$ | – | 0.0030 | – | 0.0047 |
| | | – | (0.0761) | – | (0.0883) |

Table 3: Scale-Location-Shift Model: Bias and RMSE of Estimators ($n = 100$)

| | | QAR | QADL | OLS | ADL |
|---|---|---|---|---|---|
| Normal | $\alpha$ | 0.1050 | $-0.0007$ | 0.1007 | $-0.0022$ |
| | | (0.1182) | (0.0192) | (0.1073) | (0.0170) |
| | $\beta_1$ | – | $-0.0002$ | – | $-0.0007$ |
| | | – | (0.0629) | – | (0.0335) |
| | $\beta_2$ | – | 0.0002 | – | 0.0011 |
| | | – | (0.0381) | – | (0.0327) |
| $t_3$ | $\alpha$ | 0.0372 | $-0.0018$ | 0.0789 | $-0.0062$ |
| | | (0.0565) | (0.0229) | (0.0801) | (0.0380) |
| | $\beta_1$ | – | 0.0007 | – | 0.0036 |
| | | – | (0.0626) | – | (0.1584) |
| | $\beta_2$ | – | 0.0015 | – | 0.0066 |
| | | – | (0.0640) | – | (0.1595) |
| $\chi_3$ | $\alpha$ | 0.0699 | $-0.0045$ | 0.0706 | $-0.067$ |
| | | (0.0910) | (0.0363) | (0.0809) | (0.0303) |
| | $\beta_1$ | – | 0.0003 | – | 0.0039 |
| | | – | (0.0722) | – | (0.0894) |
| | $\beta_2$ | – | 0.0040 | – | 0.0066 |
| | | – | (0.0752) | – | (0.0931) |

Table 4: Scale-Location-Shift Model: Bias and RMSE of Estimators ($n = 100$)

equation (4). The procedure proposed by Powell (1986) entails a choice of bandwidth. We consider the default bandwidth suggested by Bofinger (1975)

$$h_n = [\Phi^{-1}(\tau + c_n) - \Phi^{-1}(\tau - c_n)] \min(\hat{\sigma_1}, \hat{\sigma_2})$$

where the bandwidth $c_n = O(n^{1/3})$, $\hat{\sigma_1} = \sqrt{Var(\hat{u})}$, and $\hat{\sigma_2} = (\hat{Q}(\hat{u}, .75) - \hat{Q}(\hat{u}, .25))/1.34$. We also use a Gaussian bandwidth, but present results for the first choice of bandwidth only. The results for $n = 100$ in the experiments are presented. The results are presented in Figures 1 and 2 for $\alpha$ and $\beta_1$.

Figure 1 shows the finite sample size and power for the estimated $\alpha$ and $\beta$ coefficients considering Normal distributions and QADL and ADL estimators. Part 1 of the figure concerns $\alpha$ and Part 2 shows the results for $\beta$. Observe that the size is very close to the established five percent for all estimators. When comparing QADL and ADL estimators with respect to the Normal distribution one can see that the least squares based estimators perform better than the quantile regression estimator in terms of power under Gaussian conditions.

Figure 2 presents the results for finite sample size and power for the estimated $\alpha$ and $\beta$ coefficients considering $t_3$ distribution and QADL and ADL estimators. As in the previous case, the size is very close to the established five percent for all estimators. When the noise in the model comes from a heavier distribution, $t_3$, the QADL estimators have a superior performance *vis-a-vis* the ADL estimators, showing that there are large gains in power by using a robust estimator in the case of a non-Gaussian heavy tail distribution.

In summary, the results for the power curves show that the ADL presents more power than QADL in the Normal case, but in the $t_3$ case the inverse occurs such that QADL estimators have more power than ADL. The QADL presents a higher power *vis-a-vis* the ADL estimator in the non-Gaussian case. In addition, as expected, the comparison of same estimators using different distributions show that quantile regression based estimator performs better in a

Figure 1: Power Function for Normal innovations

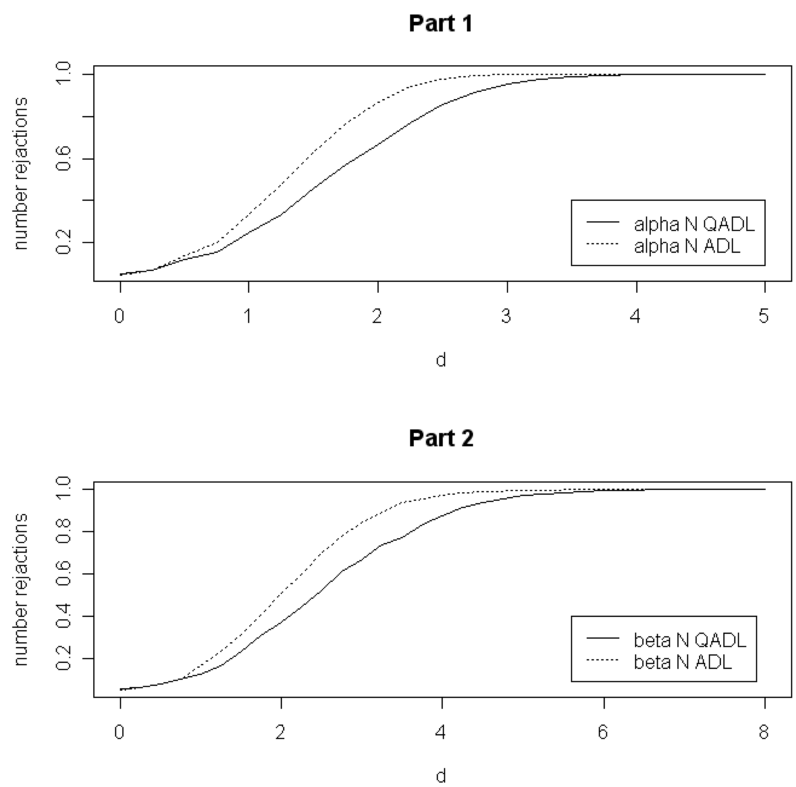
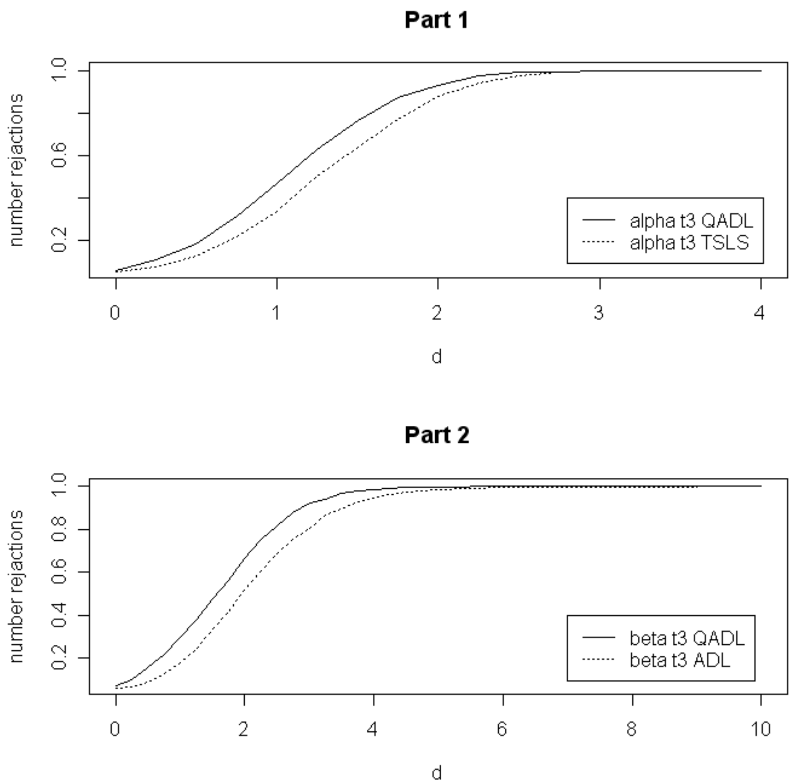


Figure 2: Power Function for t3 innovations




$t_3$ distribution case, and ADL estimator has more power under Gaussian conditions. The results for the other sample cases are qualitatively similar to those of Figures 1 and 2, but also show that, as the sample sizes increase, the tests do have improved power properties, corroborating the asymptotic theory.

In this section we also conduct Monte Carlo experiments to examine the QADL based inference procedures, where we are particularly interested in models displaying asymmetric dynamics. Thus, we consider the QADL model to test the hypothesis that $\alpha(\tau) =$ constant over $\tau$.

The data in these experiments were generated from model (12) in the same manner as in Section 3.1, where $u_t$ are $i.i.d.$ random variables. We consider the Kolmogorov-Smirnov test $KS\mathcal{W}_n$ given by (11) for different sample sizes and innovation distributions, and choose $\mathcal{T} =$

|  | Model | Normal | $t_3$ |
|---|---|---|---|
| Size | $\alpha = 0.45$ | 0.045 | 0.055 |
|  | $\alpha = 0.75$ | 0.038 | 0.052 |
| Power | $\varphi_1$ | 0.21 | 0.26 |
|  | $\varphi_2$ | 0.39 | 0.47 |

Table 5: Size and Power for Normal

[0.1, 0.9]. Both Normal innovations and $t$-Student innovations are considered. The number of repetitions is 2000.

Representative results of the empirical size and power of the proposed tests are reported in Tables 5. We report the empirical size of this test for two choices of $\alpha(\tau)$: (1) $\alpha = 0.45$; (2) $\alpha = 0.75$. For models under the alternative, we consider the following two choices:

$$\alpha = \varphi_1(u_t) \begin{cases} 0.35, u_t \geq 0, \\ 0.55, u_t < 0, \end{cases}$$

$$\alpha = \varphi_2(u_t) \begin{cases} 0.15, u_t \geq 0, \\ 0.75, u_t < 0. \end{cases}$$

Table 5 reports the empirical size and power for the case with Gaussian innovations and sample size $n = 100$, as well as the results for the $t$-Student innovations (with 3 degrees of freedom) and same sample size. Results in Table 5 show that the size of the test is close to the 5% that was set also and confirm that, using the quantile regression based approach, power gain can be obtained in the presence of heavy-tailed disturbances. (Such gains obviously depend on choosing quantiles at which there is sufficient conditional density.)

# 6 Application: House Price Returns

There is an extensive literature on cross-sectional and time-series variation in house prices, but this literature is marked by poor predictability. Mankiew and Weil (1989) find that the Baby Boom had a large impact on the US housing market. By 1989 they predicted a future slow down in the house market, which was not observed[6]. In fact, house prices have shown unprecedented values over the past 10 years. Increasing house prices had also importance in the UK. The issue of affordable housing had claimed an increasing importance in the public debate and the uncertainty about future prices is a concern of both policy makers and researchers.[7] Moreover, the fact that housing is a major component of wealth (Banks and Tanner, 2002, show that real state accounted for 35% of aggregate household wealth in the UK in the 1990s) and risky assets determines that house price changes have significant effects on aggregate consumption (see for instance Campbell and Cocco, 2007).

The evolution of house prices was extensively studied in the UK by Muellbauer and Murphy (1997), Ortalo-Magné and Rady (1999, 2006) and Rosenthal (2006) among others. Those authors rigorously studied the booms and busts in the UK housing market until 2000. In the past 50 years, there have been three major booms in the UK's owner-occupied housing market: in the early 1970s, in the late 1980s and the current housing boom. There were also smaller booms in the 1960s and, more briefly, in the late 1970s, while the early 1990s saw a bust on an unprecedented scale. Many factors conspired to produce the house price boom of the late 1980s. Initial debt levels were low as were real house prices, giving scope for rises in both. Income growth after the early 1980s recession was strong, as were income growth expectations and these became more important as a result of financial liberalization, though partly offset by bigger real interest rate effects. Wealth to income ratios grew and illiquid assets increased enhanced by financial liberalization. Financial liberalization also permitted higher gearing levels. Demographic trends were favorable with stronger population growth in the key house buying age group. The supply of houses grew more slowly, with construction of social housing falling to a small fraction of its level in the 1970s. Finally, in 1987-8 interest

[6] "Our estimates suggest that real housing prices will fall substantially - indeed, real housing prices may well reach levels lower than those experienced at any time in the past forty years." (p.236)

[7] 'Our forecasts have not been for dramatic falls in house prices, but who knows?... anyone who thinks that they can forecast asset prices is kidding themselves'. Rachel Lomax, Governor of Bank of England, The Guardian, London, November 23, 2003, cited in Rosenthal (2006, p. 289).

rates fell and the proposed abolition of property taxes in favor of the Poll Tax gave a further impetus to valuations.

The bust in the early 1990s was the result of the reversal of most of these factors. Interest rates rose from 1988-90. The bust coincided with a general recession. Demographic trends reversed. The revolt against the Poll Tax resulted in a new property tax, the Council Tax, being reintroduced. Debt levels and real house prices had reached very high levels, while wealth to income ratios then fell and recently experienced rates of return became negative and made households more cautious. Mortgage lenders tightened up their lending criteria, in a partial reversal of financial liberalization. Under these conditions, not even the major falls in nominal interest rates that took place in the early 1990s, while real interest rates remained high, were sufficient to revive UK house prices. However, the late 1990's and the new millennium showed an unprecedented increase in house prices, mostly concentrated in the Southeast (i.e. London).

The conditional quantiles provide a complete picture of the distribution of house returns conditional on past values. High quantiles correspond to unusually high returns, which can be read as a boom; low quantiles correspond to busts in the market. We propose the application of QADL to model house price returns in order to study the asymmetric behavior of this time series. We are particularly interested in the autoregressive behavior of this series at different quantiles, as well as the response from income shocks and the interest rate.

House price series are obtained from Nationwide mortgage data. Nationwide Building Society has a long history of recording and analyzing house price data and has published average house price information since 1952 while the quarterly data used here started in 1973. It is the 4th largest mortgage lender in the UK by stock and therefore its data is representative of the whole house market in the UK. The series used in this application is the average price of a representative house, UK Quarterly Index. This series is constructed by Nationwide using mortgages that are at the approvals stage and after the corresponding building survey has been completed. Approvals data is used as opposed to mortgage completions since it should give an earlier indication of current trends in prices in the residential housing market. In addition, properties that are not typical and may distort the series are also removed from the data set. The index controls for: location in the UK, type of neighborhood, floor size, property design (detached house, semi-detached house, terraced house, bungalow, flat, etc.), tenure (freehold/leasehold/feudal, except for flats, which are nearly all leasehold), number of bathrooms (1 or more than 1), type of central heating (full, part or none), type of garage (single garage, double garage or none), number of bedrooms (1,2,3,4 or more than 4), and whether property is new or not.

The series in levels are shown in figure 3. Nominal prices provide a quick overview of the magnitude of the increase in house prices. With an average value of £25,000 in 1975, the latest estimate is close to £200,000. Even when adjusting by inflation, the recent increments are significant. The current boom in house prices can be seen by the continuous growth in the past 12 years. Overall, all series show a similar performance in terms of business cycle patterns. The series show three different cycles over the past 35 years with pikes in 1980, 1990 and possibly in 2007. Interest rates are currently at a record low. Real house price returns and real GDP growth series are shown in figure 4.

In the long run, we expect that house price variations depend on its past values and some key economic variables. Based on Muellbauer and Murphy (1997) we propose an autoregressive specification of quarterly house price returns in UK using the quantile autoregressive distributed lag model. As additional covariates we use the Bank of England interest rates, real GDP growth and dummy variables for quarter effects. The proposed model is given by

Figure 3: Series

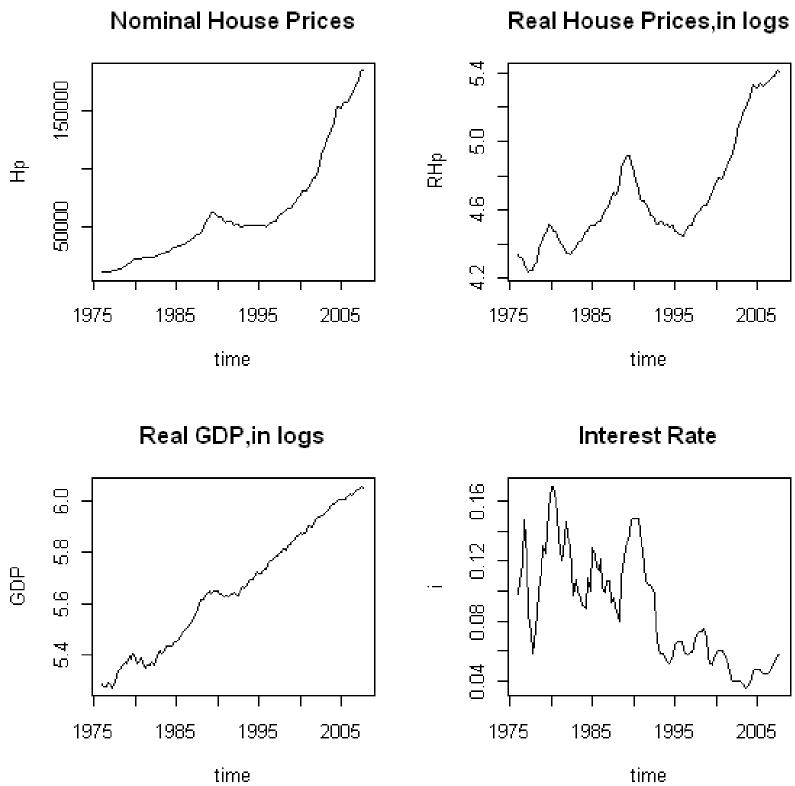
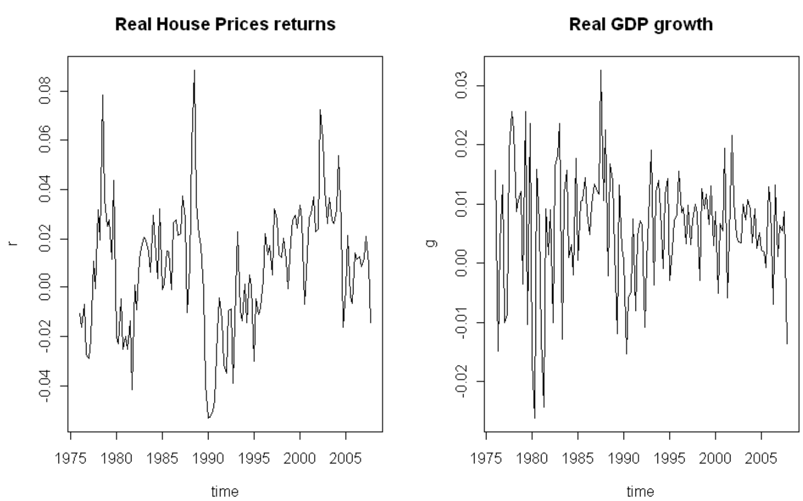



Figure 4: Series

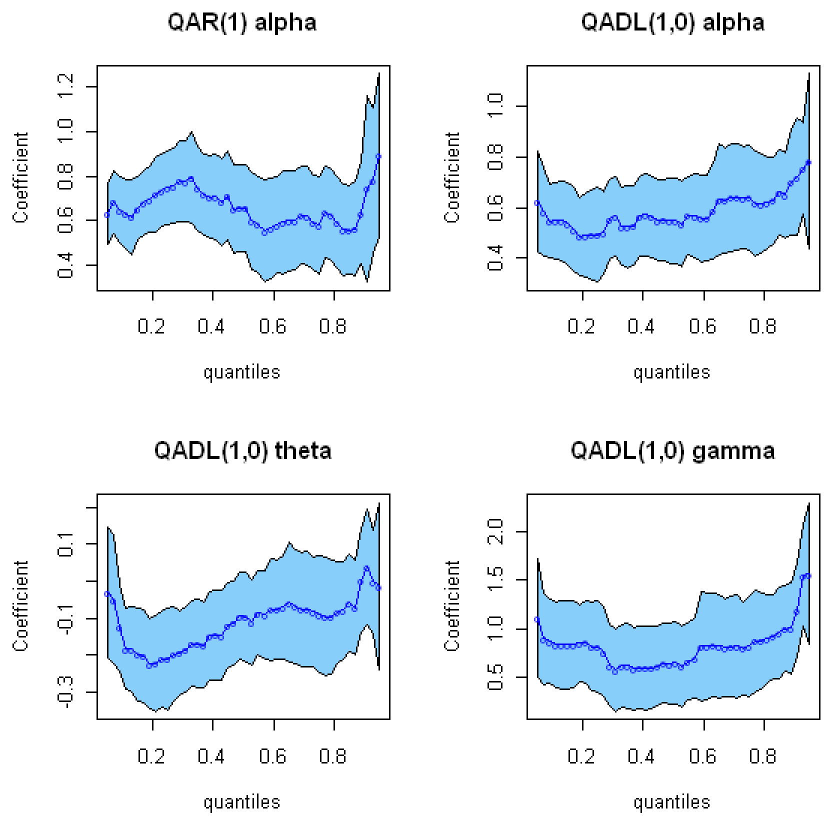


$$
\begin{aligned}
Q_{r_t}(\tau|\Im_t) \quad = \quad & \mu(\tau) + \sum_{j=1}^{p} \alpha_j(\tau) r_{t-j} + \sum_{k=0}^{q_1} \gamma_k(\tau) g_{t-k} \\
& + \sum_{l=0}^{q_2} \theta_l(\tau) i_{t-l} + \beta_1(\tau) D_{1,t} + \beta_2(\tau) D_{2,t} + \beta_3(\tau) D_{3,t},
\end{aligned} \tag{14}
$$

where, $r_t$ is the real quarterly price return in period $t$, obtained as the difference in the natural logarithm of house prices (deflated by the consumer price index), $g_t$ is the growth rate of real GDP, $i_t$ is the interest rates, and $D$ represent dummy variables for quarter effects. Note that when we exclude the covariates and the quarter dummy variables, we have back the QAR model. Augmented Dickey-Fuller tests are applied to these variables to check for unit roots. The $r_t$ series has an ADF value of -3.91 with a corresponding p-value for the null hypothesis of unit root of 0.012. $i_t$ has an ADF value of -3.70 with a p-value of 0.026. $g_t$ has an ADF value of -3.89 with a p-value of 0.017. Therefore, for all the variables we reject the unit root null hypothesis.

We first estimate the QAR model using Koenker and Xiao (2006a) methodology. We use the BIC criteria as developed in Machado (1993) for $\tau = 0.5$ to determine the number of lags to be considered, and this suggests using a QAR model with $p = 1$. This is in line with Rosenthal (2006) findings that suggest that 2 to 3-month lags are enough to model monthly house prices. Although not reported, we also apply the BIC criteria for a range of $\tau \in [0.05, 0.95]$, and in general, the model with one lag is selected, which determines that the selection for the median may be appropriate for the whole distribution. Next, we perform the QAR estimation for several quantiles and the results appear in figure 5 (QAR(1) alpha), that plots the coefficient estimates with $\pm 1.96$ times the standard error confidence interval. The results show a strong asymmetry in the lag response. Unit-root like behavior is observed for high quantiles. Overall, the QAR(1) process is globally stationary, which corroborates the use of Assumption A3.

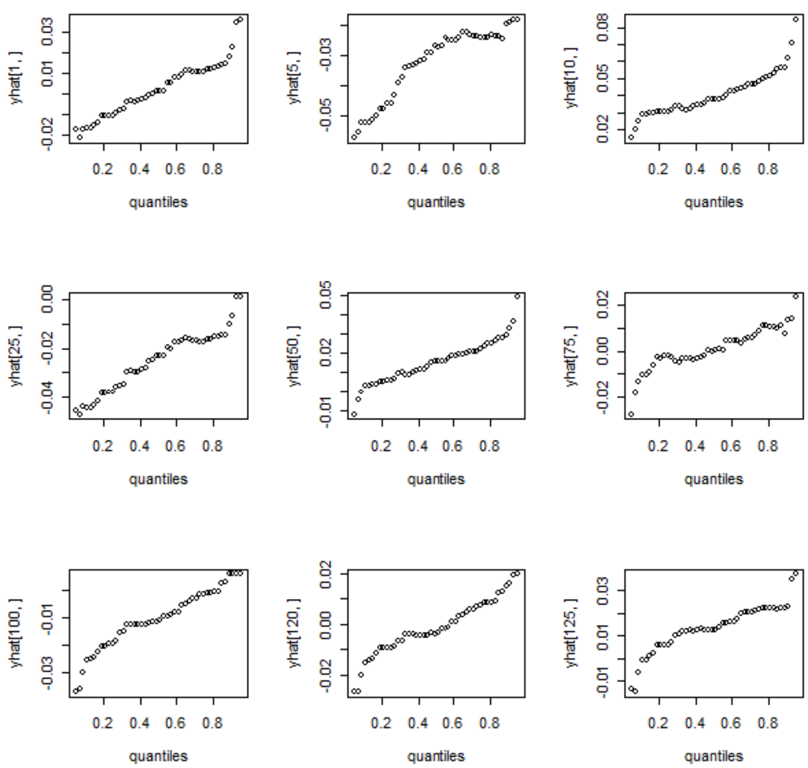
Figure 5: Coefficient estimates




Next, we consider our suggested QADL model. Applying a similar BIC criteria we select $p = 1$, $q_1 = 0$ for GDP, i.e. the contemporaneous effect of GDP, and $q_2 = 0$ for the interest rate. For the latter, we have a smaller BIC for the interest rate lagged one quarter than with the contemporaneous value. For this reason we consider the interest rate lagged one quarter only. The estimates shown in figure 5 (QADL(1,0) alpha) suggests that the model still shows unit-root-like behavior only in the high extreme quantiles. That is, the model seems to show global stationarity with some persistence in unusually high shocks. Note that the inclusion of the covariates determines a more homogeneous increasing behavior of the $\alpha$-coefficients along different quantiles than that observed in the QAR model.

The interest rate has a negative impact on house price returns, although only statistically significant for low quantiles (see figure 5, QADL(1,0) theta). In other words, this variable

Figure 6: Predicted Quantiles



may have an effect to prevent busts, but it may not deter house price booms. Therefore, the policy followed by the Bank of England of cutting the interest rate to prevent a house price collapse may have the desired effect. A Kolmogorov-Smirnov test of the hypothesis that $\sup_{\tau \in \mathcal{T}} \theta_1(\tau) = 0$ gives a KS value of 12.8. Looking at Andrews (1993, p.840) the critical values are 8.19, 9.84, 13.01 for 10%, 5% and 1% significance levels respectively. Therefore, the interest rate has an effect different from zero at the 5% significance level.

Real GDP growth has a larger impact on low and high quantiles than for medium quantiles (see figure 5, QADL(1,0) gamma). For low quantiles, this is interpreted as the fact that GDP growth reactivates the housing market when returns are low, while it might be contributing to house prices' busts (as that in the early 1990's). Moreover, it contributes to sustaining house prices increments. In other words, periods of unusually high returns are very responsive to GDP growth. Note that the estimated coefficient for very high quantiles is greater than 1, although not statistically different from this value except for a few quantiles. Poterba (1991) and Capozza, Hendershott, Mack, and Mayer (2002) among others, provide evidence on the asymmetric responses of house prices to income shocks and overshooting. The QADL estimates present this overshooting feature of house prices to income shocks, but restricted to high quantiles. A Kolmogorov-Smirnov test of the hypothesis that $\sup_{\tau \in \mathcal{T}} \gamma_0(\tau) = 0$ gives a KS value of 33.1, which by the critical values discussed above show that the effect of GDP is not zero (as expected from the figure). However, the hypothesis that $\sup_{\tau \in \mathcal{T}} \gamma_0(\tau) = 1$ gives KS=4.1. Then, overall, the effect of GDP growth on house price returns is not different from 1, and there is no overshooting in the quantile process.

One important assumption of the QADL model is monotonicity, discussed in Section 2. In order to check for this condition, we evaluate the monotonicity by predicting in-sample quantiles for several observations. These are plotted in figure 6. In general, the predicted values are increasing in $\tau$, which corroborates that our specification is valid.

In summary, the application illustrates the usefulness of the QADL process to model asymmetric behavior in time series. Of particular importance are the asymmetries in the slope of the lagged dependent variable and other covariates in both extreme low and high quantiles. In this case, the conditional mean may be a misleading estimator in periods of extremely low and high returns, which are those when policymakers are keener to intervene or to predict future behavior.

# 7 Conclusion

We have developed a quantile autoregression distributed lag model (QADL), and discussed a dynamic specification in the quantile autoregression framework. Quantile regression methods provide a framework for robust estimation and inference and allow one to explore a range of conditional quantiles exposing a variety of forms of conditional heterogeneity under less compelling distributional assumptions. The model is able to accommodate exogenous covariates in the quantile autoregression model. We show that the estimators are consistent and asymptotically normal. In addition, we suggest a Wald and Kolmogorov-Smirnov (KS) type tests for general linear hypotheses.

Monte Carlo studies are conducted to evaluate the finite sample properties of the proposed QADL estimator for several types of distributions. It is shown that the estimator proposed by Koenker and Xiao (2006a) is severely biased by omitting exogenous variables, while the QADL is generally unbiased. In addition, the QADL approach has a better performance *vis-a-vis* ordinary augmented distributed lag (ADL) approach in terms of the root mean square error of the estimators for non-Gaussian heavy tail distributions. We also investigate the size and power of the test statistics comparing QADL with ADL and the results show that there are large power gains in using tests based on QADL especially when innovations are heavy-tailed.

We illustrate the QADL model with an application to quarterly house price returns data in the UK. The results show that house price returns have an asymmetric autoregressive behavior, and that real GDP growth and interest rates have an asymmetric impact on house prices variations along the quantiles. In addition, the results suggest that unit root behavior is present only in the high extreme quantiles. Thus, the model seems to show global stationarity with some persistence in unusually high returns. The inclusion of covariates determines a more homogeneous increasing behavior of the autoregressive coefficients along different quantiles than that observed in the QAR model, but maintains the persistence in the high quantiles. The interest rate has a negative impact on house prices, mostly significant for low quantiles. This can be interpreted as the fact that the interest rates have an effect on stimulating the demand in the real estate market when returns are low, but it does not deter house prices booms. In addition, there is evidence that the impact of GDP on house prices presents an asymmetric persistence and it is stronger for low and high quantiles. For low quantiles, this is interpreted as the fact that GDP growth reactivates the real estate market when returns are low, while it might be contributing to house prices' busts. Moreover, it contributes to sustaining house prices booms. In other words, periods of unusually high returns are very responsive to GDP growth. In fact, there is some evidence of overshooting for high quantiles.

# 8 Appendix: Proofs

## 8.1 Proof of Theorem 1

Following Koenker and Xiao (2006a) we denote $E(\alpha_{j,t}) = \mu_j$, $E[\theta_{l,t}] = m_l$, and assume that $1 - \sum_{j=1}^{p} \mu_j \neq 0$. Let $\mu_y = (\mu_0 + \mu_x \sum_{l=0}^{q} m_l)/(1 - \sum_{j=1}^{p} \mu_j)$, where $\mu_x = E[x_t]$, and denote

$$\underline{y}_t = y_t - \mu_y \quad \text{and} \quad \underline{x}_t = x_t - \mu_x.$$

We have

$$\underline{y}_t = \alpha_{1,t}\underline{y}_{t-1} + \cdots + \alpha_{p,t}\underline{y}_{t-p} + \theta_{0,t}\underline{x}_t + \cdots + \theta_{q,t}\underline{x}_{t-q} + \upsilon_t, \tag{15}$$

where

$$\upsilon_t = u_t + \mu_y \sum_{l=1}^{p}(\alpha_{l,t} - \mu_l) + \mu_x \sum_{l=0}^{q}(\theta_{l,t} - m_l).$$

It is easy to check that $E\upsilon_t = 0$ and $E\upsilon_t\upsilon_s = 0$ for any $t \neq s$ since $Eu_t = 0$, $E\alpha_{l,t} = \mu_l$, $E\theta_{l,t} = m_l$ and $u_t$ is *iid*. We have to find an $\mathcal{F}_t$-measurable solution for (15) to derive stationarity condition for the process $\underline{y}_t$. We define the $p \times 1$ random vectors $\underline{Y}_t = [\underline{y}_t, \underline{y}_{t-1}, \cdots, \underline{y}_{t-p+1}]'$ and $V_t = [\upsilon_t, 0, \cdots, 0]'$, and $p \times p$ random matrices

$$A_t = \begin{bmatrix} A_{p-1,t} & \alpha_{p,t} \\ \mathbf{I}_{p-1} & \mathbf{0}_{(p-1)\times 1} \end{bmatrix} \quad \text{and} \quad \Theta_t = \begin{bmatrix} \Theta_{q-1,t} & \theta_{q,t} \\ \mathbf{0}_{(p-1)\times(q-1)} & \mathbf{0}_{(p-1)\times 1} \end{bmatrix},$$

where $A_{p-1,t} = [\alpha_{1,t}, \alpha_{2,t}, \cdots, \alpha_{p-1,t}]$, $\Theta_{q-1,t} = [\theta_{0,t}, \theta_{1,t}, \cdots, \theta_{q-1,t}]$. Then,

$$E[V_t V_t'] = \begin{bmatrix} \sigma_\upsilon^2 & \mathbf{0}_{1\times(p-1)} \\ \mathbf{0}_{(p-1)\times 1} & \mathbf{0}_{(p-1)\times(p-1)} \end{bmatrix} = \Sigma,$$

and the original process can be written as

$$\underline{Y}_t = A_t\underline{Y}_{t-1} + \Theta_t\underline{X}_t + V_t.$$

By substituting $\underline{Y}_{t-1}$ recursively, we have

$$\underline{Y}_t = \underline{Y}_{t,m} + \underline{X}_{t,m} + R_{t,m},$$

where

$$\underline{Y}_{t,m} = \sum_{j=0}^{m} B_j V_{t-j}, \quad R_{t,m} = B_{m+1}\underline{Y}_{t-m-1}, \quad \underline{X}_{t,m} = \sum_{j=0}^{m} B_j \Theta_{t-j}\underline{X}_{t-j},$$

and

$$B_j = \begin{cases} \prod_{l=0}^{j-1} A_{t-l} & j \geq 1 \\ \mathbf{I} & j = 0. \end{cases}$$

The stationarity of an $\mathcal{F}_t$-measurable solution for $y_t$ involves the convergence of $\{\sum_{j=0}^{m} B_j V_{t-j}\}$, $\{R_{t,m}\}$ and $\{\underline{X}_{t,m}\}$ as $m$ increases for fixed $t$. We need to verify that $\mathrm{vec}E(\underline{Y}_t\underline{Y}_t^T)$ converges as $m \to \infty$. Let $V = E(\underline{Y}_t \underline{Y}_t')$ then

$$\begin{aligned}
\text{vec}V &= \text{vec}E(\underline{Y}_t\underline{Y}_t^T) = \text{vec}E[(\underline{Y}_{t,m} + \underline{X}_{t,m} + R_{t,m})(\underline{Y}_{t,m} + \underline{X}_{t,m} + R_{t,m})^T] \\
&= \text{vec}[E(\underline{Y}_{t,m}\underline{Y}_{t,m}^T) + E(\underline{Y}_{t,m}\underline{X}_{t,m}^T) + E(\underline{Y}_{t,m}R_{t,m}^T) + E(\underline{X}_{t,m}\underline{Y}_{t,m}^T) + E(\underline{X}_{t,m}\underline{X}_{t,m}^T) \\
&\quad + E(\underline{X}_{t,m}R_{t,m}^T) + E(R_{t,m}\underline{Y}_{t,m}^T) + E(R_{t,m}\underline{X}_{t,m}^T) + E(R_{t,m}R_{t,m}^T)] \qquad (16)
\end{aligned}$$

Notice that $B_j$ is independent with $V_{t-j}$ and $\{u_t, t = 0, \pm 1, \pm 2, \cdots\}$ are independent random variables, thus, $\{B_j V_{t-j}\}_{j=0}^{\infty}$ is an orthogonal sequence in the sense that $E[B_j V_{t-j} B_k V_{t-k}] = 0$ for any $j \neq k$. Thus,

$$\begin{aligned}
\text{vec}E[\underline{Y}_{t,m}\underline{Y}_{t,m}^T] &= \text{vec}E\left[(\sum_{j=0}^{m} B_j V_{t-j})(\sum_{j=0}^{m} B_j V_{t-j})^T\right] \\
&= \text{vec}E\left[\sum_{j=0}^{m} B_j V_{t-j} V_{t-j}^T B_j^T\right] \\
&= \sum_{j=0}^{m} \prod_{l=0}^{j-1} E(A_{t-l} \otimes A_{t-l}) \text{vec}E(V_{t-j}V_{t-j}^T)
\end{aligned}$$

since $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$ and $(\prod_{k=0}^{j} A_k) \otimes (\prod_{k=0}^{j} B_k) = \prod_{k=0}^{j}(A_k \otimes B_k)$. If we denote

$$A = E[A_t] = \begin{bmatrix} \bar{\alpha} & \alpha_p \\ \mathbf{I}_{p-1} & \mathbf{0}_{(p-1)\times 1} \end{bmatrix},$$

where $\bar{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_{s-1}]$, , then $A_t = A + \Xi_t$, where $E(\Xi_t) = 0$, and

$$E[A_{t-l} \otimes A_{t-l}] = E[(A + \Xi_t) \otimes (A + \Xi_t)] = A \otimes A + E(\Xi_t \otimes \Xi_t) = \Omega_A.$$

Thus,

$$\text{vec}E\left[(\sum_{j=0}^{m} B_j V_{t-j})(\sum_{j=0}^{m} B_j V_{t-j})^T\right] = \sum_{j=0}^{m} \Omega_A^j \text{vec}(\Sigma).$$

Similarly, notice that $B_j$ is independent with $\Theta_{t-j}$, and $\underline{Y}_{t-j-1}$, $V_{t-j}$ and $\Theta_{t-j}$ are independent for $j = 1, 2, \cdots, m$. Thus, each expectation in the RHS of (16) are calculated by

$$\begin{aligned}
\mathrm{vec}E[\underline{Y}_{t,m}\underline{X}_{t,m}^T] &= \sum_{j=0}^{m} \Omega_A^j E[[\Theta_{t-j}^T \otimes V_{t-j}]\mathrm{vec}(\underline{X}_{t-j}^T)], \\
\mathrm{vec}E[\underline{X}_{t.m}\underline{Y}_{t,m}^T] &= \sum_{j=0}^{m} \Omega_A^j E[[V_{t-j}^T \otimes \Theta_{t-j}]\mathrm{vec}(\underline{X}_{t-j})], \\
\mathrm{vec}E[R_{t,m}R_{t,m}^T] &= \Omega_A^{m+1}\mathrm{vec}V \\
\mathrm{vec}E[\underline{Y}_{t,m}\underline{Y}_{t,m}^T] &= \sum_{j=0}^{m} \Omega_A^j \mathrm{vec}(\Sigma), \\
\mathrm{vec}E[\underline{X}_{t,m}\underline{X}_{t,m}^T] &= \sum_{j=0}^{m} \Omega_A^j \Omega_\Theta \mathrm{vec}E[\underline{X}_{t-j}\underline{X}_{t-j}^T],
\end{aligned}$$

where $\Omega_\Theta = E[\Theta_{t-j} \otimes \Theta_{t-j}]$, and the other four terms are equal to zero. The matrix $\Omega_A$ can be represented in Jordan canonical form as $\Omega_A = P\Lambda P^{-1}$, where $\Lambda$ has the eigenvalues of $\Omega_A$ along its main diagonal. If the eigenvalues of $\Omega_A$ have moduli less than unity, $\Lambda^j$ converges to zero at a geometric rate. Notice that $\Omega_A^j = P\Lambda^j P^{-1}$, following similar analysis as Nicholls and Quinn (1982) and Koenker and Xiao (2006a), $\underline{Y}_t$ is stationary.

## 8.2 Proof of Theorem 2

Consistency of the estimator can be achieved by the argmax theorem in van der Vaart and Wellner (1996). We first state a lemma which will help to prove the theorem.

**Lemma 1** *Under assumptions A1-A6*

$$\sup_{\beta \in \mathscr{A} \times \mathscr{B}} |g_n(z_t, \beta) - E[g(z_t, \beta)]| = o_p(1)$$

*where* $g_n \equiv \frac{1}{n}\sum_{i=1}^{n} \rho_\tau(y_t - z_t'\beta)$, *and* $g \equiv \rho_\tau(y_t - z_t'\beta)$.

**Proof.** Let

$$\mathscr{P} = \{g_n(\beta) : \beta \in \Theta = \mathscr{A} \times \mathscr{G}\}$$

and note that by assumption A5 $\Theta$ is compact, as argued in Lemma B2 of Chernozhukov and Hansen (2006), $\mathscr{P}$ is continuous and uniformly Lipschitz over $\Theta$. Therefore by Lemma 3.10 in van de Geer (2000) we have that $H_{1,B}(\delta, \mathscr{P}, P) < \infty$, that is, the $\delta$-entropy with bracketing of $\mathscr{P}$ is finite. Hence, it satisfies a uniform law of large numbers, and the lemma follows. ■

Now we prove Theorem 2.

**Proof.** By bounded density function condition A2 and A5-A6, $g(\cdot)$ is continuous over $\mathscr{A} \times \mathscr{B}$. By Lemma 1 $\sup_{\beta \in \mathscr{A} \times \mathscr{B}} |g_n(z_t, \beta) - E[g(z_t, \beta)]| \xrightarrow{p} 0$. This implies by Corollary 3.2.3 in van der Vaart and Wellner (1996) that $\hat{\beta}(\tau) \xrightarrow{p} \beta(\tau)$. ■

## 8.3 Proof of Theorem 3

The next two lemmas help in the derivation of the results. The first lemma is an application of Theorem 1 and standard central limit theorem. The second lemma is only a combination of results.

**Lemma 2** *Under Assumptions A1-A6,*

$$n^{-1/2} \sum_{t=1}^{n} z_t \psi_\tau(u_{t\tau}) \Rightarrow \Omega_0,$$

*where*

$$\Omega_0 = \begin{bmatrix} 1 & \mu_y & \mu_x \\ \mu_y & \Omega_y & \Omega_{yx} \\ \mu_x & \Omega_{yx} & \Omega_x \end{bmatrix}$$

$$\Omega_y = \begin{bmatrix} \gamma_{y_0} & \cdots & \gamma_{y_{p-1}} \\ \vdots & \ddots & \vdots \\ \gamma_{y_{p-1}} & \cdots & \gamma_{y_0} \end{bmatrix}, \ \Omega_x = \begin{bmatrix} \gamma_{x_0} & \cdots & \gamma_{x_{q-1}} \\ \vdots & \ddots & \vdots \\ \gamma_{x_{q-1}} & \cdots & \gamma_{x_0} \end{bmatrix}, \ \Omega_{yx} = \begin{bmatrix} \gamma_{yx0} & \cdots & \gamma_{yp-1} \\ \vdots & \ddots & \vdots \\ \gamma_{yp-1} & \cdots & \gamma_{y0} \end{bmatrix},$$

*with* $E(y_t) = \mu_y$, $E(x_t) = \mu_x$, $E(y_t y_{t-j}) = \gamma_{y_j}$, $E(x_t x_{t-j}) = \gamma_{x_j}$,$E(y_t x_{t-j}) = \gamma_{yx_j}$.

**Proof.** By definition of $u_{t\tau}$, assumptions A1-A6, and Theorem 1, we have that $E\left[\psi_\tau(u_{t\tau})|\Im_t\right] = 0$, $z_t\psi_\tau(u_{t\tau})$ is a martingale difference sequence and so $n^{-1/2}\sum_{t=1}^n z_t\psi_\tau(u_{t\tau})$ satisfies a central limit theorem. As it is well know in the quantile regression literature, following the arguments of Portnoy (1984) and Gutenbrunner and Jureckova (1992), the autoregression quantile process is tight and thus the limiting variate viewed as a function of $\tau$ is a Brownian Bridge over $\tau \in T$.

$$n^{-1/2}\sum_{t=1}^n z_t\psi_\tau(u_{tr}) \Rightarrow \Omega_0^{1/2}B_k(\tau).$$

For a fixed $\tau$, $n^{-1/2}\sum_{t=1}^n z_t\psi_\tau(u_{t\tau})$ converge to a $(p+q)$-dimensional vector normal variate with covariance matrix $\tau(1-\tau)\Omega_0$. ■

**Lemma 3** *Under Assumptions A1-A6,*

$$\sum_{t=1}^n \int_0^{(n^{-1/2}v)'z_t} \{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\}ds \Rightarrow \frac{1}{2}v'\Omega_1 v,$$

*where* $\Omega_1 = \lim n^{-1}\sum_{t=1}^n f_{t-1}[F_{t-1}^{-1}(\tau)]z_t z_t'$.

**Proof.** Consider the limiting distribution of $W_n(v) = \sum_{t=1}^n \int_0^{(n^{-1/2}v)'z_t}\{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\}ds$, write it as

$$W_n(v) = \sum_{t=1}^n \xi_t(v), \quad \xi_t(v) = \int_0^{(n^{-1/2}v)'z_t} \{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\}ds.$$

Further, define $\bar{\xi}_t(v) = E\{\xi_t(v)|\Im_t\}$, and $\bar{W}_n(v) = \sum_{t=1}^n \bar{\xi}_t(v)$, then, as in Koenker and Xiao (2006a), $\{\xi_t(v) - \bar{\xi}_t(v)\}$ is martingale difference sequence. Note that

$$u_{t\tau} = y_t - z_t\beta(\tau) = y_t - F_{t-1}^{-1}(\tau),$$

and

$$
\begin{aligned}
\bar{W}_n(v) &= \sum_{t=1}^{n} E\left\{\int_0^{(n^{-1/2}v)^T z_t} [I(u_{t\tau} \le s) - I(u_{t\tau} < 0)]ds|\Im_{t-1}\right\} \\
&= \sum_{t=1}^{n} \int_0^{(n^{-1/2}v)^T z_t} \left[\int_{F_{t-1}^{-1}(\tau)}^{s+F_{t-1}^{-1}(\tau)} f_{t-1}(r)dr\right] ds \\
&= \sum_{t=1}^{n} \int_0^{(n^{-1/2}v)^T z_t} \left[\frac{F_{t-1}(s+F_{t-1}^{-1}(\tau)) - F_{t-1}(F_{t-1}^{-1}(\tau))}{s}\right] sds.
\end{aligned}
$$

Under Assumption A6

$$
\begin{aligned}
\bar{W}_n(v) &= \sum_{t=1}^{n} \int_0^{(n^{-1/2}v)^T z_t} f_{t-1}(F_{t-1}^{-1}(\tau))sds + o_p(1) \\
&= \frac{1}{2n}\sum_{t=1}^{n} f_{t-1}(F_{t-1}^{-1}(\tau))v'z_t z_t'v + o_p(1).
\end{aligned}
$$

Let $\Omega_1 = \lim n^{-1}\sum_{t=1}^{n} f_{t-1}[F_{t-1}^{-1}(\tau)]z_t z_t'$, so,

$$
\bar{W}_n(v) \Rightarrow \frac{1}{2}v'\Omega_1 v.
$$

Using the argument as Herce (1996), the limiting distribution of $\sum_{t=1}^{n}\xi_t(v)$ is the same as $\sum_{t=1}^{n}\bar{\xi}_t(v)$. Therefore

$$
\sum_{t=1}^{n}\int_0^{(n^{-1/2}v)'z_t}\{I(u_{t\tau} \le s) - I(u_{t\tau} < 0)\}ds \Rightarrow \frac{1}{2}v'\Omega_1 v.
$$

■

Now we prove Theorem 3.

**Proof.** As discussed in the text, the limiting distribution of $\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau))$ is giving by the two terms of $H_n(v)$, say $n^{-1/2}\sum_{t=1}^{n} v'z_t\psi_\tau(u_{t\tau})$ and $\sum_{t=1}^{n}\int_0^{(n^{-1/2}v)'z_t}\{I(u_{t\tau} \le s) - I(u_{t\tau} < 0)\}ds$. Thus, once we show that $H_n(\cdot)$ converge weakly to $H(\cdot)$ we just need to find the minimizer of $H(\cdot)$, and $\hat{v} = \sqrt{n}(\hat{\beta}(\tau) - \beta(\tau))$ converges in distribution to that minimizer.

Denote $\psi_\tau(u) = \tau - I(u < 0)$ for $u \neq 0$. Following the approach of Knight (1989) and using the identity

$$\begin{aligned} \rho_\tau(u-v) - \rho_\tau(u) &= -v\psi_\tau(u) + (u-v)\{I(0 > u > v) - I(0 < u < v)\} \\ &= -v\psi_\tau(u) + \int_0^v \{I(u \leq s) - I(u < 0)\}ds \end{aligned} \tag{17}$$

the objective function for minimization of problem (7) can be rewritten as

$$\begin{aligned} H_n(v) &= \sum_{t=1}^{n}[\rho_\tau(u_{t\tau} - (n^{-1/2}v)'z_t - \rho_\tau(u_{t\tau})] \\ &= -\sum_{t=1}^{n} v'n^{-1/2}z_t\psi_\tau(u_{t\tau}) + \sum_{t=1}^{n}\int_0^{(n^{-1/2}v)^T z_t} \{I(u_{t\tau} \leq s) - I(u_{t\tau} < 0)\}ds. \end{aligned}$$

Now, by Lemma 2 and Lemma 3 we have that

$$H_n(v) \Rightarrow -v'\Omega_0^{1/2}B_k(\tau) + \frac{1}{2}v'\Omega_1 v = H(v).$$

Finally, by convexity Lemma of Pollard (1991) and arguments of Knight (1989), note that $H_n(v)$ and $H(v)$ are minimized at $\hat{v} = \sqrt{n}(\hat{\beta}(\tau) - \beta(\tau))$ and $\Sigma^{1/2}B_k(\tau)$ respectively, where and $\Sigma = \Omega_1^{-1}\Omega_0\Omega_1^{-1}$. By Lemma A of Knight (1989) we have,

$$\Sigma^{-1/2}\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \Rightarrow B_k(\tau).$$

■

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