α-quantile Option in a Jump-Diffusion Economy

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Abstract

In this note, we extend the analysis of the behaviour of the α-quantile option to the case of a contract's underlying security driven by a Lévy process. To this aim, a simulation procedure based on the order statistics is implemented. The results produced are also used to study the connections between the occurring of a jump in the market and option prices. In particular, we show that, no matter the risk-neutral valuation framework chosen, the occurring of a jump affects the tails of the distribution of the functional which defines the option payoff. Since options pay a premium for the probability mass existing in the tails of such a distribution, this fact might be seen as a first key to interpret the observed biases.

Key words: Lookback option, α-quantile option, Lévy processes, Lévy-Khintchine formula, incomplete markets, order stastistic.

1 Introduction

Lookback options represent a typology of path-dependent contracts with payoff determined by the maximum or the minimum price of the underlying asset within the life of the option. In particular, a fixed-strike lookback call pays the highest value achieved by the stock price during the contract lifetime; hence it can be considered by an investor as an insurance against large downward movements of the stock near maturity, and might be thought of as a way to deal with the market exit problem. Analogously, the lookback put provides a protection from substantial rises in the market near expiration. However, these features make lookback options too expensive and hence not attractive to ordinary investors. To overcome this limitation new types of exotic contracts came to existence in the attempt of reducing the option price whilst preserving its potential payoff. Examples of new lookback-type

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contracts are the partial lookback options introduced by Heynen and Kat (1994), which are characterized by a monitoring period for the extreme value of the underlying asset price to be a subset of the option's lifetime; and the fractional lookback, for which only a percentage of the extreme values is in effect in the payoff function of the options.

Another possible alternative contract could be identified in the α-quantile option introduced by Miura (1992). An α-quantile call option with strike K and underlying asset S has a payoff function at maturity T defined as $\left(S_0 e^{Q(\alpha,T)} - K\right)^+$, where S_0 is the value of the underlying asset at the beginning of the contract and $Q(\alpha, T)$ is the α-quantile of the process X, driving the underlying asset price S. In particular Q is defined to be the smallest level below which the process spends at least a fraction $\alpha \in (0,1)$ of some period $[0,T]$, that is

$$Q(\alpha, T) = \inf\left\{x : \int_0^T 1_{(X_t \leq x)} dt > \alpha T\right\}.$$

It follows from the definition that

$$\lim_{\alpha \to 1} Q(\alpha, T) = \sup_{0 \leq t \leq T} X_t$$

and

$$\lim_{\alpha \to 0} Q(\alpha, T) = \inf_{0 \leq t \leq T} X_t.$$

Using this property, Ballotta and Kyprianou (2000) have shown that the α-quantile option is comparatively cheaper than the fixed strike lookback written on the same underlying and with monitoring period equal to the contract lifetime. Precisely this feature suggests an interesting potential use of this path-dependent contract, introduced mainly as a "mathematical exercise" and not yet traded in the market. In fact, it might be seen as a valid tool to generate at maturity returns similar to the lookback option ones but for a less expensive initial investment. Since the convergence of the α-quantile option price to the price of the equivalent lookback, the investor has also the possibility to increase and control the leverage effect of his portfolio in a quite flexible way through a suitable setting of the parameter α.

Close pricing formulas for the α-quantile option have been derived in the Brownian motion setting by both Akahori (1995) and Dassios (1995). However, these valuation formulas present serious computational difficulties because they are still expressed in integral form. Ballotta and Kyprianou (2000) have implemented a numerical valuation procedure that removes some of these problems by taking advantage of the Dassios-Port-Wendel identity, which expresses the distribution of the α-quantile of a Brownian motion as the convolution of the supremum and the infimum of the process itself.

The aim of this communication is to analyze the behaviour of the α-quantile option in a more realistic setup for the market model. Precisely, we will consider a general Lévy motion as relevant process for the price of the underlying security and we will derive the price of the α-quantile option in such a framework. We will also introduce a Monte Carlo simulation procedure for this price and, since a jump-diffusion economy identifies in general an incomplete market, such a procedure will be extended to price the option under different risk-neutral martingale measures. Particular attention will be given to the mispricing generated by the misspecification of a jump-diffusion process for the underlying asset as a pure diffusion process.

The rest of the paper proceeds as follows: Section 2 develops the market setup and the α-quantile option pricing model. In Section 3, we present the simulation procedure and discuss the numerical evidence produced. Concluding remarks are offered in Section 4. Proof of the pricing equation is provided in the Appendix.

2 α-quantile option and Lévy processes

In this section we will analyze more in detail the behaviour of the α-quantile option when the underlying asset is driven by a general Lévy motion. In order to provide a complete presentation, we need first to specify the market model used.

2.1 The market model

Let $\{L_t : t \geq 0\}$ be a Lévy process and consider a frictionless market with continuous trading. Assume that there are no taxes, no transaction costs, no restrictions on borrowing or short sales and all securities are perfectly divisible. Assume further that only two securities are traded: the money market account $B_t = e^{rt}$, $r > 0$, and a non-dividend paying risky asset, S, whose value at time $t \geq 0$ is $S_t = S_0 e^{L_t}$, $S_0 > 0$. From the Lévy-Khintchine formula, it follows that the dynamic of the stock price, under some risk-neutral equivalent probability measure $\mathbb{P}^*$ and for some $\sigma \in \mathbb{R}^+$, is given by the following

$$dS_t = [r - \lambda \mathbb{E}^*(z)] S_{t\_} dt + \sigma S_{t\_} dW_t^* + S_{t\_} \int_{\mathbb{R}} z N(dt, dz) \tag{1}$$

where:

- r is the risk-free interest rate,
- W_t^* is a standard one-dimensional $\mathbb{P}^*$-Brownian motion,
- $N(dt, dz)$ is a homogeneous Poisson counting measure on $\mathbb{R} \times \mathbb{R}^+ \setminus \{0\}$, of rate λ, with compensator $\nu(dt, dz) = \nu(dz)\, dt$ on $\mathbb{R} \times \mathbb{R}^+ \setminus \{0\}$, and independent of W_t^*,

- z is the proportion of the stock jump and it is assumed to be such that $z = e^x - 1$, where X is modelled through a sequence of i.i.d. random variables with mean μ_X and variance σ_X^2; X is also independent of W_t^* and N, and it represents the jump size of the Lévy process driving the stock price. If $f(dz)$ is the density function of z, then $\nu(dz) = \lambda f(dz)$.

The corresponding stochastic differential equation for the driving Lévy process is consequently given by

$$dL_t = \left[r - \frac{\sigma^2}{2} - \lambda \mathbb{E}^*(z)\right] dt + \sigma dW_t^* + \int_{\mathbb{R}} xN(dt, dx);$$

in virtue of the assumptions previously discussed and setting ν^* to be the $\mathbb{P}^*$-compensator of N, we can rewrite this last equation as follows

$$dL_t = \left[r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} z\nu^*(dz)\right] dt + \sigma dW_t^* + \int_{\mathbb{R}} xN(dt, dx). \tag{2}$$

As mentioned before, the setup defined by equation (1) is an incomplete market, meaning that there exists at least one contingent claim which cannot be hedged and hence that, under the assumption of no arbitrage, there is a multiplicity of equivalent martingale measures, $\mathbb{P}^*$, under which agents do evaluate the risk. This implies that the risk-neutral drift of the Lévy process we observe in equation (2), i.e.

$$\beta^* = r - \frac{\sigma^2}{2} - \lambda \mathbb{E}^*(z) = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} z\nu^*(dz)$$

cannot be univocally specified.

In the following of this paper we are going to price the α-quantile option under alternative risk-neutral paradigms in order to examine how the option can be affected by choosing a specific scheme for risk pricing. In particular, we are going to consider the Esscher martingale measure with parameter θ (Gerber and Shiu, 1994), the minimal martingale measure (Föllmer and Schweizer, 1991) and the martingale measure underlying the model proposed by Merton (1976) for option pricing when the jump risk is uncorrelated with the market. The drift characterizing equation (2) in each of the above mentioned approaches is reported in Table 1. For more details about the derivation of the drift formulas, we refer the reader to Ballotta (2000). What can be observed here is that this risk-neutral drift is given by two components: the standard risk-neutral drift of the Brownian motion which characterizes the Black-Scholes model, and the expected value (computed under some martingale measure $\mathbb{P}^*$) of the proportion of the variation in the stock price caused by the occurring of a jump in the market. The only exception is represented by the drift under the Merton measure, since this expectation is computed using the

"Risk-neutral" martingale measure	β
Merton measure: $\mathbb{P}^M$	$\beta^M = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} z\nu(dz)$
Esscher measure: $\mathbb{P}^\theta$	$\beta^\theta = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} e^{-\theta x}(e^x - 1)\nu(dx)$
Minimal measure: $\mathbb{P}^m$	$\beta^m = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (bx+1)(e^x - 1)\nu(dx)$

Table 1: Drift of the Lvy process under different risk-neutral valuation framework.

real probability measure $\mathbb{P}$. This is due to the fact that in his model Merton (1976) assumes the jump risk to be asset specific, and hence diversifiable. Which implies that no premium is paid for such a risk. Therefore the change of measure does affect only the Brownian motion component of the Lévy process, leaving the Poisson part unchanged.

2.2 The α-quantile option

Let us consider the market framework defined by equation (1). Define $Q(\alpha, T)$ to be the α-quantile of the Lévy process L, i.e. the process such that

$$Q(\alpha, T) = \inf\left\{ l : \int_0^T 1_{(L_t \leq l)} dt > \alpha T \right\}.$$

Applying the risk-neutral valuation procedure, we can say that the no-arbitrage price at time $t \in [0, T]$ of an α-quantile call option, paying $\left(S_0 e^{Q(\alpha,T)} - K\right)^+$ at maturity, is given by

$$C(S_0, \alpha, T-t) = e^{-r(T-t)} \mathbb{E}^* \left[\left(S_0 e^{Q(\alpha,T)} - K\right)^+ \mid \mathcal{F}_t \right], \tag{3}$$

where $\mathbb{E}^*$ denotes the expectation under some equivalent risk-neutral martingale measure $\mathbb{P}^*$. Then the following result holds:

Proposition 1 *The price at time $t \in [0, T]$ of an α-quantile call option equals*

$$\begin{aligned} C \quad & (S_0, \alpha, T-t) \\ = \quad & e^{-r(T-t)} \int_K^\infty \mathbb{P}^* \left[Q'(\alpha', T-t) > \ln \frac{z}{S_t} \right] 1_{\left(\int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds > t - (1-\alpha) T \right)} dz \\ & + e^{-r(T-t)} \int_K^\infty 1_{\left(\int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds \leq t - (1-\alpha) T \right)} dz, \end{aligned} \tag{4}$$

where

$$\alpha' = \frac{\alpha T - \int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds}{T - t}$$

and $Q'(.,.)$ is a version of the α-quantile which is independent of $\mathcal{F}_t$ and such that

$$Q'(\alpha', T-t) \stackrel{d}{=} \sup_{0 \le \tau \le \alpha'(T-t)} L_\tau + \inf_{0 \le \tau \le (1-\alpha')(T-t)} L_\tau. \tag{5}$$

Equation (5) is the generalization due to Dassios (1996) to the case of a process with independent and stationary increments, like a Lévy motion, of the Dassios-Port-Wendel identity (Dassios, 1995). The proof of Proposition 1 follows the same steps as in Dassios (1995) for the case of a Brownian motion-driven underlying asset, and it is reported in Appendix A.

The pricing formula expressed by equation (4) presents serious computational difficulties arising from the fact that all the occupation time sets $\left\{ \int_0^t 1_{\left(L_s \le \ln \frac{z}{S_0}\right)} : z \ge K \right\}$ have to be recorded, the price is still expressed in integral form and the distribution of the α-quantile process is known only in its convoluted form. Hence, the need to implement a numerical pricing procedure, which will be described in the next section.

3 Pricing the α-quantile option in a jump-diffusion economy

For the case of an underlying asset driven by an arithmetic Brownian motion, a simulation procedure for the α-quantile option price which exploits the Dassios-Port-Wendel identity and the distributions of the extremes of the Brownian motion has been implemented by Ballotta and Kyprianou (2000). Such a procedure is based on a Monte Carlo technique where the quantile $Q(\alpha, T)$ is generated as the sum of two independent samples of the extremes of the Brownian motion with drift. Despite the fact the Dassios-Port-Wendel identity holds also for processes with stationary and independent increments, the extension of this numerical scheme to the Lévy process case cannot be based on the same methodology because there are no expressions for the extremes' distributions of a Lévy motion easy to handle in practice. In the present note we use a standard Monte Carlo procedure in which the paths of the Lévy process are generated as a sequence of partial sums. This allows to obtain the α-quantile value via its discrete-time analogue, the order statistics, which can be defined as follows.

Definition 2 *Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables and S be the sequence of their successive partial sums, i.e. $S = (S_0, S_1, ...)$ with $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, $\forall n = 1, 2, ...$ Rearrange the sequence of partial sums $S_0, S_1, ..., S_n$ in increasing order $S_{(0)} \le S_{(1)} \le ... \le S_{(n)}$, where $(0), (1), ..., (n)$ is a permutation of*

$0, 1, ..., n$. *The new variables*

$$Q_{0:n}(S) = S_{(0)}; Q_{1:n}(S) = S_{(1)}; ...; Q_{n:n}(S) = S_{(n)}$$

are called the order statistics of a sample of size $n+1$. *More precisely,*

$$Q_{j:n}(S) = \inf \left\{ s : \sum_{i=0}^{n} 1_{(S_i \leq s)} > j \right\}$$

is the (j, n)*-th order statistic of* S, $\forall j \in \{0, 1, ..., n\}$.

Therefore, according to this definition, $Q_{0:n}(S) = \min_{0 \leq i \leq n} S_i$ and $Q_{n:n}(S) = \max_{0 \leq i \leq n} S_i$. In other words, the quantile of a process defined as the sequence of partial sums arranges the sums in increasing order and the (j, n)-th quantile, $Q_{j:n}(S)$, is the sum which is j-th from the bottom according to the given order. More in detail, we subdivide the option contract lifetime $[0, T]$ into n equal subintervals of length $\Delta t = \frac{T}{n}$, and we define $t_j = \frac{j}{n}T$, $j = 1, 2, ..., n$. Since the occurring of a jump is governed by a Poisson process of rate λ, for each subinterval we sample a random number R from a uniform distribution $\mathcal{U}(0, 1)$, which allows to define an auxiliary random variable I_j such that $I_j = 1$ if $R \leq \lambda \Delta t$, that is if a jump occurs in $(t_{j-1}, t_j]$, and $I_j = 0$ otherwise. In this way, assuming that the jump size of the Lévy motion is normally distributed, that is $X \sim N(\mu_X, \sigma_X^2)$, the path of the Lévy process can be obtained by the following

$$L_{t_j} = L_{t_{j-1}} + \beta^* \Delta t + \sigma \sqrt{\Delta t} y_j + (\mu_X + z_j \sigma_X) I_j.$$

where y_j and z_j are two independent random samples from a standardized normal distribution. Once the sequence of partial sums $L_0 = L_{t_0}, L_{t_1}, L_{t_2}, ..., L_{t_n} = L_T$ is generated, it is ordered in increasing order, and the j-th value from the bottom is then selected, where $j = [\alpha n]$, in order to get the α-quantile Q_j of the process L. The final payoff is, therefore, computed as $P = \left(S_0 e^{Q_j} - K\right)^+$ and the numerical approximation to the option price at time $t = 0$ is $C := e^{-rT} \sum_{i=1}^{m} \frac{P_i}{m}$. The antithetic variable technique is then applied to all the random components involved for variance reduction purposes.

3.1 Monte Carlo simulation and results

The Monte Carlo experiment is carried out by simulating 10,000 paths, with each path composed by 2,500 steps (equivalent to 7 observation per day over one year).

Unless otherwise stated, the basic parameter set is:

$$\begin{array}{llll} K = 100; & r = 5\%; & T = 1 & \alpha = 0.5; \\ \lambda = 0.59; & \mu_X = -0.0537; & \sigma_X = 0.07. & \end{array}$$

Stock	JD ($\mathbb{P}^M$)	GBM	GBM/JD	%
90	1.5226	1.6618	1.0914	-8.38
	(0.028)	(0.0093)		
95	3.0624	3.2801	1.0711	-6.64
	(0.0364)	(0.0125)		
100	5.5861	5.7758	1.0339	-3.28
	(0.0418)	(0.0156)		
105	8.8955	9.1075	1.0238	-2.33
	(0.0406)	(0.0192)		
110	12.902	13.0363	1.0104	-1.03
	(0.0388)	(0.0226)		
115	17.3031	17.4131	1.0063	-0.63
	(0.0388)	(0.0255)		
120	21.8755	22.004	1.0058	-0.58
	(0.0384)	(0.0279)		

Table 2: Quantile Options Prices under Geometric Brownian motion (GBM) and Jump-Diffusion (JD) process, when the Merton valuation framework is used. Parameter set: $K = 100$; $r = 5\%$; $T = 1$ $\alpha = 0.5$; $\lambda = 0.59$; $\mu_X = -0.0537$; $\sigma_X = 0.07$; $\sigma_S = 0.2$. The column labeled GBM/JD represents the ratio between GBM option prices to JD option prices. The column labeled % reports the percentage variation in the option price, (JD-GBM)/GBM. The numbers in parentheses correspond to the standard errors of the Monte Carlo simulation.

Finally, we assume that the instantaneous total volatility, σ_S, of the underlying asset is constant, in order to perform a sensible comparison between the option price obtained in the Black-Scholes framework and the price deriving from the Lévy model. In particular, we fix $\sigma_S = 0.2$.

Table 2 shows prices of the α-quantile call for different stock prices obtained under the Merton risk-neutral probability measure, $\mathbb{P}^M$. It shows that the assumption of geometric Brownian motion-modeled underlyings overprices call options and this mispricing is even more accentuated for out-of-the-money contracts.

Table 3 shows the same analysis for the case of option prices obtained under the Esscher risk-neutral martingale measure, $\mathbb{P}^\theta$, and the minimal martingale measure, $\mathbb{P}^m$. Also in these cases, the mispricing appears to be particularly accentuated for out-of-the-money options, especially under the Esscher pricing framework.

A first explanation for this type of mispricing might derive from the structure of the α-quantile process itself. As seen before, it is the smallest level below which the process spends αT of its time. If on average a negative jump is expected to occur, this is likely to lower the entire process L and hence also the value of Q. This would imply a lower value of the quantile option call with respect to the price of the

Stock	JD ($\mathbb{P}^{\theta}$)	GBM/JD	%	JD ($\mathbb{P}^{m}$)	GBM/JD	%
90	1.4515	1.1449	-12.65	1.4828	1.1207	-10.77
	(0.0274)			(0.0277)		
95	3.0042	1.0918	-8.41	2.9889	1.0974	-8.88
	(0.0367)			(0.0368)		
100	5.3462	1.0804	-7.44	5.4187	1.0659	-6.18
	(0.0409)			(0.0421)		
105	8.6931	1.0477	-4.55	8.5973	1.0593	-5.60
	(0.0413)			(0.0402)		
110	12.632	1.0320	-3.10	12.6079	1.0339	-3.29
	(0.0388)			(0.0399)		
115	16.8626	1.0326	-3.16	16.8837	1.0314	-3.04
	(0.0375)			(0.0374)		
120	21.4360	1.0265	-2.58	21.4556	1.0255	-2.49
	(0.0377)			(0.0373)		

Table 3: Quantile Options Prices generated by a Jump-Diffusion (JD) process under the Esscher measure and the minimal martingale measure. Parameter set: $K = 100$; $r = 5\%$; $T = 1$ $\alpha = 0.5$; $\lambda = 0.59$; $\mu_X = -0.0537$; $\sigma_X = 0.07$; $\sigma_S = 0.2$. The column labeled GBM/JD represents the ratio between GBM option prices to JD option prices. The column labeled % reports the percentage variation in the option price, (JD-GBM)/GBM. The numbers in parentheses correspond to the standard errors of the Monte Carlo simulation.

	$\mathbb{P}^{M}$	$\mathbb{P}^{\theta}$	$\mathbb{P}^{m}$	**BS**
Mean	0.014075	0.011924	0.011761	0.013916
Variance	0.014815	0.014663	0.014683	0.014559
Skewness	-0.01422	-0.01901	-0.01955	-0.00742
Kurtosis	3.467914	3.467574	3.461758	3.437838

Table 4: Estimated moments of the 0.5-quantile process. Parameter set: $r = 5\%$; $T = 1$ $\alpha = 0.5$; $\lambda = 0.59$; $\mu_X = -0.0537$; $\sigma_X = 0.07$; $\sigma_S = 0.2$.

equivalent contract in the Brownian motion regime. In Table 4 the first four moments of the estimated 0.5-quantile distribution are reported; in particular the mean of the simulated quantile distribution seems to confirm what previously described, with the only exception of the Merton valuation framework.

A second reason for the large price discrepancy in the α-quantile options might be found looking at the other moments of the α-quantile distributions in the cited table. In fact, as the prices of the α-quantile call at time $t = 0$ can be rewritten as

$$\begin{aligned} C(S_0, \alpha, T) &= e^{-rT}\mathbb{E}^*\left[\left(S_0 e^{Q(\alpha,T)} - K\right)^+\right] \\ &= e^{-rT}\int_K^\infty \mathbb{P}^*\left[Q(\alpha, T) > \ln\frac{z}{S_0}\right] dz, \end{aligned}$$

we can say that the call contract pays a premium for the probability mass placed in the right tail of the α-quantile distribution. Hence, the coefficients of skewness and kurtosis provide important information about the direction of the mispricing. Table 4 shows that the α-quantile distribution arising in a jump-diffusion economy is more negatively skewed and with higher kurtosis than in the Brownian motion economy, which means that the Brownian motion model places more probability mass in the right tail than the Lévy process model, causing therefore the observed overpricing. This fact would also explain the reason why, as moneyness decreases, there is less price sensitivity to distributional assumptions. Moreover, Table 4 outlines that this is particularly true for the α-quantile distribution evaluated in the Esscher pricing framework. This suggests that the moments of the estimated distribution of the quantile process may be also a key to interpret the differences between the prices obtained under the Merton measure, the Esscher and the minimal martingale ones. In fact, according to Table 4, the Merton measure should provide on average the biggest call prices, i.e. the smallest mispricing for the call contract, which is what we observe comparing the results in Table 2 with the results contained in Table 3.

4 Conclusion

The aim of this work has been to study a new kind of financial instrument, the α-quantile option, introduced first by Miura (1992), which is at the moment only a theoretical object since it is not yet traded in the market. The properties of this option have been in particular presented in a general jump diffusion setting where the underlying asset is driven by a Lévy process with normally distributed jumps. A numerical pricing procedure has been implemented and the numerical evidence produced has been used to analyze the mispricing generated, and how and through which elements the introduction of jumps in the model affects financial securities prices. The attention has been given especially to the moments of the estimated payoff functional's distribution.

One problem left open is how to deal numerically with the mid-contract value of the quantile option. In fact, in this case pricing formulae and numerical approximations cannot avoid the set of occupation times needed to define the α-quantile itself and which makes this process not Markovian. Also a refinement of the numerical procedure used to simulate the option prices would be recommended. The simulation procedure, in fact, relies on the Monte Carlo technique which, for a path-dependent contract, is not very efficient since the relevant stock value could be missed. A proposal might be to try the implementation of a procedure based on the Dassios-Port-Wendel identity, via some numerical approximation for the extremes's distribution of a Lévy process, as for example the one proposed by Nahum (1998) for the running maximum of the process L. Finally, we outline that the analysis carried on in this paper concerns only the mispricing arising for different values of the stock price, all the other parameters kept constant. A more detailed study of the mechanisms through which the effects produced by the occurring of a jump in the market are spread from the underlying to the derivative security, would require an accurate comparative statics analysis.

A Proof of Proposition 1

Let us consider first some preliminary results; precisely consider the risk-neutral formula for the α-quantile option price

$$C\left(S_0, \alpha, T-t\right) = e^{-r(T-t)}\mathbb{E}^*\left[\left(S_0 e^{Q(\alpha,T)} - K\right)^+ \mid \mathcal{F}_t\right],$$

where $\mathbb{E}^*$ denotes the expectation under some equivalent risk-neutral martingale measure $\mathbb{P}^*$. The same expression can be also rewritten as

$$\begin{aligned} C\left(S_0, \alpha, T-t\right) &= e^{-r(T-t)} \int_K^\infty \mathbb{P}^*\left[S_0 e^{Q(\alpha,T)} > z \mid \mathcal{F}_t\right] dz \\ &= e^{-r(T-t)} \int_K^\infty \mathbb{P}^*\left[Q\left(\alpha, T\right) > \ln \frac{z}{S_0} \mid \mathcal{F}_t\right] dz \end{aligned} \tag{6}$$

By definition of α-quantile, we get:

$$\left\{Q\left(\alpha, T\right) > \ln \frac{z}{S_0}\right\} \Leftrightarrow \left\{\int_0^T 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds < \alpha T\right\}.$$

But

$$\begin{aligned} \left\{\int_0^T 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds < \alpha T\right\} &\Leftrightarrow \left\{\int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds + \int_t^T 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds < \alpha T\right\} \\ &\Leftrightarrow \left\{\int_t^T 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds < \alpha T - \int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds\right\}. \end{aligned}$$

Since L has independent and stationary increments, it follows that

$$\mathbb{P}^*\left[L_s \leq \ln\frac{z}{S_0} \mid \mathcal{F}_t\right] = \mathbb{P}^*\left[L_t + L'_u \leq \ln\frac{z}{S_0}\right] \quad \text{for } 0 \leq t \leq s \leq T \ \text{ and } t+u=s,$$

where L'_u is an independent copy of the Lévy process. Therefore

$$\begin{aligned}
& \mathbb{P}^*\left[\int_t^T 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds < \alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds \mid \mathcal{F}_t\right] \\
= \ & \mathbb{P}^*\left[\int_0^{T-t} 1_{\left(L'_u \leq \ln\frac{z}{S_0} - \ln\frac{S_t}{S_0}\right)} ds < \alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds\right]
\end{aligned}$$

since $L_t = \ln\frac{S_t}{S_0}$ by construction. Let

$$\alpha' = \frac{\alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds}{T-t}$$

Then, under the assumption that $\int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds < \alpha T$, when $0 < \alpha' < 1$, that is when $\int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds > t - (1-\alpha) T$, we get

$$\begin{aligned}
& \mathbb{P}^*\left[Q(\alpha, T) > \ln\frac{z}{S_0} \mid \mathcal{F}_t\right] \\
= \ & \mathbb{P}^*\left[\int_0^{T-t} 1_{\left(L'_u \leq \ln\frac{z}{S_t}\right)} ds < \alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds\right] \\
= \ & \mathbb{P}^*\left[Q'(\alpha', T-t) > \ln\frac{z}{S_t}\right] \qquad (7)
\end{aligned}$$

where $Q'(\cdot,\cdot)$ is a version of the α-quantile which is independent of $\mathcal{F}_t$. Instead, when $\int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds \leq t - (1-\alpha) T$ or, equivalently, $T - t \leq \alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds$, we observe that

$$\mathbb{P}^*\left[\int_0^{T-t} 1_{\left(L_s \leq \ln\frac{z}{S_t}\right)} ds < \alpha T - \int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds\right] = 1 \qquad (8)$$

since $\int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds \leq T - t$ and $\int_0^t 1_{\left(L_s \leq \ln\frac{z}{S_0}\right)} ds < \alpha T$.

Equations (7) and (8) allow to prove Proposition 1

Proof. At first we observe that

$$\begin{aligned}\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathit{1}_{\left(S_0e^{Q(\alpha,T)}>z\right)}\mid\mathcal{F}_t\right]\\ &= \mathbb{E}^*\Bigg[\mathit{1}_{\left(S_0e^{Q(\alpha,T)}>z\right)}\Bigg(\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)T\right)}\\ &\quad+\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}\Bigg)\,|\mathcal{F}_t\Bigg].\end{aligned}$$

Since the events $\left\{\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)\,T\right\}$ and $\left\{\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)\,T\right\}$ are $\mathcal{F}_t$-measurable by definition of the occupation time, then

$$\begin{aligned}\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right] &= \mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)T\right)}\\ &\quad+\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}.\end{aligned}$$

For $\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds<\alpha T$, in virtue of equations (7) and (8) we get

$$\begin{aligned}&\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)T\right)}\\ =\;&\mathbb{P}^*\left[Q'\left(\alpha',T-t\right)>\ln\frac{z}{S_t}\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)T\right)}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}\\ =\;&\mathbb{P}^*\left[\int_0^{T-t}\mathit{1}_{\left(L_s\leq\ln\frac{z}{S_t}\right)}ds<\alpha T-\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}\\ =\;&\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{P}^*\left[S_0e^{Q(\alpha,T)}>z\mid\mathcal{F}_t\right]\\ =\;&\mathbb{P}^*\left[Q'\left(\alpha',T-t\right)>\ln\frac{z}{S_t}\right]\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds>t-(1-\alpha)T\right)}+\mathit{1}_{\left(\int_0^t \mathit{1}_{\left(L_s\leq\ln\frac{z}{S_0}\right)}ds\leq t-(1-\alpha)T\right)}\end{aligned}\tag{9}$$

Whilst, if $\int_0^t 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds \geq \alpha T$, $\mathbb{P}^* \left[S_0 e^{Q(\alpha,T)} > z \mid \mathcal{F}_t\right] = 0$ because the events

$$\left\{Q(\alpha,T) > \ln \frac{z}{S_0}\right\} \Leftrightarrow \left\{\int_0^T 1_{\left(L_s \leq \ln \frac{z}{S_0}\right)} ds < \alpha T\right\}$$

are not compatible with the assumption done. The result (4) follows by substitution of (9) in (6). ∎

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