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Glide paths for a retirement plan with deferred annuities

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We construct investment glide paths for a retirement plan using both traditional asset classes and deferred annuities. The glide paths are approximated by averaging the asset proportions of stochastic optimal investment solutions. The objective function consists of power utility in terms of secured retirement income from purchased deferred annuities, as well as a bequest that can be withdrawn before retirement. Compared with conventional glide paths and investment strategies, our deferred annuity-enhanced glide paths provide the investor with higher welfare gains, more efficient investment portfolios, and more responsive retirement income patterns and bequest levels to different fee structures and personal preferences.

Keywords: Retirement planning, deferred annuity, glide path strategy, multi-stage stochastic programming.

Glide paths for a retirement plan with deferred annuities

Investment strategies that incorporate "glide paths" are widely used for pension planning. Asset classes can be roughly divided into two categories: a risky asset class, such as equity, and a less risky asset class, such as developed economy government bonds. The glide path investment strategy typically reduces the equity proportion and increases the bond proportion in the investment portfolio as an individual approaches retirement. However, these strategies do not typically include deferred annuities as an asset class, even though an annuity is normally an important source of a retirees' income. Annuity markets have started to give investors the freedom to use deferred annuities. U.S. insurance companies began selling deferred income annuities (DIA) in 2011. Currently more than sixteen insurers offer the product (Chen et al., 2019). LIMRA estimate that total annuity sales in 2019 were $241.7 billion while deferred income annuity sales were a much smaller $2.5 billion. Academic research has paid relatively little attention to how these products can be integrated into a pre-retirement investment portfolio. The US Treasury allows target date funds to include deferred annuities among their assets in 401(k) plans (United States Department of Treasury, 2014). Many retirement funds, however, apply more traditional, simple glide path strategies where the allocation to the risky asset class declines as a linear function of the individual's age. In doing so they fail to maximize an investor's utility in terms of retirement income (Merton, 2014).

Starting with a stochastic optimal investment solution to a retirement planning problem, we calculate glide paths that include traditional asset classes – cash, bonds, and equities – but which also incorporate deferred annuities. The objective function that we specify maximizes the expected value of a sum of time-separable power utility (constant relative risk aversion) functions expressed in terms of a secured retirement income at retirement and a bequest before retirement. In order to solve the optimization problem, we employ multi-stage stochastic programming (MSP) which is widely used in operations research (Ziemba, 2003). We then implement a strategy that averages the proportions of the optimal investment and deferred annuity allocations over time as our new glide path strategy.[1] In particular, we extend the approach of Konicz et al. (2016), but we focus on the accumulation phase.

There are a significant number of papers that explore the optimal choice involving immediate fixed, variable and inflation-linked annuities. Most studies focus on various types of annuity strategies only at or after retirement (i.e., in a decumulation phase). Koijen et al. (2011) present the optimal full-annuitization portfolio of fixed, inflation-indexed and variable annuities with changes in state values upon retirement and its hedging strategy in the pre-retirement period. In the hedging strategy, the optimal composition of nominal and inflation-linked bonds depends upon the annuity strategy that will be used at retirement. Also, Boulier et al. (2001), Deelstra et al. (2003) and Cairns et al. (2006) show the optimal dynamic asset allocations for enhancing the utility of lifetime consumption when there is hedging demand for annuity risk.

There are fewer studies, however, that focus on the optimal investment and deferred annuity choice for an individual prior to retirement (Horneff et al., 2010; Maurer et al., 2013;

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[1] For more examples of MSP applications to individual retirement planning, see Consigli et al. (2012), Dempster and Medova (2011), and Konicz et al. (2016).

Huang et al., 2017). They generally assume that: the individual has an uncertain labour income stream; that the individual invests their disposal income in risk-free bonds and risky stocks; and that the individual is able to buy deferred annuities at any time before retirement. The deferred annuities begin to pay lifetime benefits at a fixed retirement date. The individual may also have state or defined-benefit-type pension entitlements, which pay lifetime retirement income as some function of their final salary. These studies show that deferred annuities have a crucial role in increasing welfare gains. The optimal strategy is to start purchasing deferred annuities early (from age 40) and to continue to purchase them over time until they comprise about 80 percent of the final portfolio at retirement (Horneff et al., 2010; Maurer et al., 2013).

Huang et al. (2017) consider purchases of deferred annuities where the interest rate process is mean-reverting. Deferred annuities are, of course, expensive in the current low interest rate environment. If the interest rate process is mean-reverting, Huang et al. (2017) show that a risk-neutral person should wait until the yield reaches the long-term average before buying deferred annuities. For a risk-averse investor, there exists an entry boundary for interest rates, at which the investor begins to buy deferred annuities. There is also an exit boundary at which the investor spends all of their remaining wealth to buy deferred annuities. Huang et al. (2017) provide an asymptotic approximation for the boundary strategy. However, no investment uncertainty and additional cash inflow is assumed, that is, there is no capital growth and no contributions from labour income.

The optimal timing for annuitization is another important decision in retirement planning. Horneff et al. (2008, 2009) investigate optimal dynamic annuitization and investment choices on immediate constant-payment annuities during the retirement period and with immediate variable-payment annuities as well as allocations to stocks and bonds prior to retirement. Buying an increasing quantity of immediate annuities enhances the individual's welfare. The optimal equity allocation over time declines, which corresponds to the typical life-cycle pattern; the optimal bond allocation increases over time. Blake et al. (2003) examine the performance of distribution strategies using constant-payment annuities, equity-linked annuities and equity-linked income-drawdown. They note that a higher bequest motive and a larger fund size lead to a later annuitization age. Without the bequest motive, the optimal annuitization age is very sensitive to relative risk aversion. For a highly risk-averse individual, immediate annuitization is optimal.

In this paper, we adopt the general framework of these earlier studies, but we focus here on the appropriate glide-path strategy for a personal retirement plan where deferred annuity purchases are available in the accumulation phase. Using multi-stage stochastic programming (MSP), we find an optimal solution to the optimization problem and use average proportions of the optimal investment and deferred annuity allocations over time as the glide path strategy. The investment opportunities are time-varying. Compared with simulation and dynamic programming which are widely used for solving retirement planning problems numerically, MSP enables us to incorporate sophisticated financial market models and realistic constraints, such as those on asset classes, transaction costs and taxes.

We investigate the performance of our glide paths with different fee structures and personal preferences by comparing these strategies with conventional retirement-plan strategies, such as a constant-mix, glide-path and "$100 - age$" investment strategies. Our main contribution to the retirement planning literature in this paper is the introduction of a new, optimally-based glide path, that incorporates deferred annuities in the accumulation phase. Our results show that this new approach is superior to traditional glide paths in terms expected retirement income per unit risk. Many retirement funds currently apply simple glide path strategies that do not allow for the incorporation of deferred annuities. Also, the traditional glide path strategies fail to maximize an investor's utility in terms of retirement income.

The rest of this article is organized as follows. We describe the portfolio optimization problem, the price dynamics of available assets and the construction of our new glide paths with deferred annuities in the Section entitled 'Investment for a Retirement Plan'. In the Section entitled 'Financial Modelling and Data', the time-varying and predictable market movements of equity returns and yield curves are defined using a vector autoregressive model with the Nelson-Siegel model. We solve the model by applying a multi-stage stochastic programming approach. Descriptions of the model formulation are given in the Section entitled 'Multi-stage Stochastic Programming Formulation'. We investigate the numerical results of stochastic optimal solutions produced by the multi-stage stochastic programming approach and by the new glide paths in the Section entitled 'Results'.

Investment for a Retirement Plan

The investment problem. We consider an individual investor who has a personal retirement plan at time 0, who is δ years of age and who retires at time T. During the retirement planning period $[0, T)$ they contribute a fixed proportion ϕ of their labour income L_t (at time t) every year to the retirement plan. The individual can hold cash, bond and equity funds in the retirement plan, which is worth W_t at time t and can make withdrawals. The individual can only buy deferred annuities which will pay out at time T (if the individual survives until this date) every year from time T until the individual's death. In our model the annuities are irreversible contracts, which means that the individual can only purchase annuities. Every unit of annuity that is bought pays out a secured income of £1 annually in retirement. If the individual dies before retirement, then the annuities do not pay out and are terminated, but the wealth in their fund is bequeathed to the individual's heirs. If they survive until retirement, then the accumulated wealth in the plan is fully annuitized to purchase an immediate annuity.

During the retirement planning period $[0, T)$, the individual selects an asset allocation, including deferred annuities, in order to maximize the expected utility of retirement income and of the bequest before retirement. Note that we consider investment for a retirement plan only, and therefore we assume that the individual can separate the utility from retirement income from the utility derived from pre-retirement consumption.

The individual investor has a power utility function $u(t, x) = e^{-\rho t} x^{1-\gamma}/(1-\gamma)$ in terms of cash flow or wealth x at time t. Their utility therefore has a constant relative risk aversion (CRRA) parameter γ. As γ tends to one, the utility function becomes log utility. We assume that all retirement income is used for consumption, so that the utility function is defined in terms of the income generated by the annuities. The time impatience parameter $0 \leq \rho \leq 1$ reflects the individual's preference for early cash flow compared to late cash flow. The utility function is also defined with regards to the bequest amount W_t before retirement.

We adopt standard, actuarial notation to represent survival and death probabilities. The probability that a δ-year-old person survives until age $\delta + t$ is denoted as ${}_t p_\delta$. The probability that a person aged $(\delta + t)$ years dies over the following Δt years is denoted as ${}_{\Delta t} q_{\delta+t}$, abbreviated to $q_{\delta+t}$ when $\Delta t = 1$. For practical purposes, we also assume that a person cannot live beyond age ω, which is the maximum age in an actuarial life table, so the individual investor dies before or at time $\tau = \omega - \delta$, since they are δ years old at time 0.

Let the total number of units of deferred annuities purchased by time t be $X_{A,t}$, where the subscript A stands for annuities. Since each unit of annuity provides £1 annually during retirement, the secured retirement income by time t is $X_{A,t}$. If the annuity price is $S_{A,t}$, then the investor pays $S_{A,t}(X_{A,t} - X_{A,t-1})$ to buy annuities at time $t \in [0, T]$ (with $X_{A,-1} = 0$).

We also assume that the investor buys and sells units or shares in a cash, bond or an equity fund, denoted C, B, and E respectively .[2] Let $X_{B,t}$ be the number of units of the bond fund held in the retirement plan at time t, and $S_{B,t}$ be the price of bond units at time t. A corresponding notation holds for the cash and equity funds. At time t, the individual decides how much to hold in cash, bonds, and equities, and how many annuity units to buy. The decision variable for the individual at time $t \in [0, T)$ is therefore $X_t = [X_{C,t}, X_{B,t}, X_{E,t}, X_{A,t}]'$.

The retirement planning problem consists of the objective function, budget constraints, and variable constraints given by the equation below:

$$\max_{X_t, t\in[0,T)} \quad \mathbb{E}_0\left[\sum_{t\in[T,\tau)} {}_t p_\delta u(t, X_{A,T}) + \sum_{t\in[0,T)} {}_t p_\delta \cdot q_{\delta+t} \cdot \kappa^\gamma u(t+1, W_{t+1})\right], \tag{1a}$$

$$s.t. \quad W_{t+1} = W_t + \phi \cdot L_t - S_{A,t}(X_{A,t} - X_{A,t-1}) + \textstyle\sum_{i\in\{C,B,E\}}(S_{i,t+1} - S_{i,t})X_{i,t} \quad \text{for } t \in [0, T), \tag{1b}$$

$$X_{A,t} \geq X_{A,t-1} \quad \text{for } t \in [0, T], \tag{1c}$$

$$X_{A,T} = X_{A,T-1} + W_T/S_{A,T}, \tag{1d}$$

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[2] We can think of these funds as being mutual funds.

$$X_{i,t} \geq 0 \qquad \text{for } i \in \{C, B, E, A\} \text{ and } t \in [0, T), \tag{1e}$$

$$X_{i,T} = 0 \qquad \text{for } i \in \{C, B, E\}, \tag{1f}$$

$$W_t \geq 0 \qquad \text{for } t \in [0, T], \tag{1g}$$

$$W_0 = w_0, \; X_{A,-1} = 0 \quad \text{w.p. 1.} \tag{1h}$$

In Equation (1a) above, the decision variables over which expected utility is maximized are the portfolio and annuity purchase decisions over the planning horizon $[0, T)$. Since the retirement income secured through deferred annuity purchases by retirement time T is $X_{A,T}$, the utility of secured income at time $t \in [T, \tau)$ during retirement is $u(t, X_{A,T}) = e^{-\rho t}(X_{A,T})^{1-\gamma}/(1-\gamma)$. A bequest parameter κ captures the importance of bequest relative to retirement income. If the individual dies during period $[t, t+1)$, then wealth W_{t+1} constitutes a bequest, so that the utility of the bequest is then $\kappa^\gamma \cdot u(t+1, W_{t+1}) = \kappa^\gamma \cdot e^{-\rho(t+1)}(W_{t+1})^{1-\gamma}/(1-\gamma)$.

The budget constraint, in Equation (1b) above, shows the dynamics of wealth W_t in the retirement plan. Wealth is increased by a contribution which is a fixed proportion ϕ of labour income L_t, as well as by changes in the price of the cash, bond, and equity funds, $(S_{i,t+1} - S_{i,t})$ for $i = \{C, B, E\}$. Wealth in the retirement plan is reduced if there is a withdrawal of $S_{A,t}(X_{A,t} - X_{A,t-1})$ to buy deferred annuities at time t.

The constraint in Equation (1c) means that annuities can be bought but not sold, while the constraint in Equation (1e) means that short sales are not allowed. The terminal conditions at retirement time T in Equations (1d) and (1f) assert that cash, bond, and equity holdings are sold, and all wealth in the retirement plan is annuitized. Equation (1g) ensures that wealth remains positive. The initial conditions in Equation (1h) state that the investor has a known initial wealth and no deferred annuity at time 0.

The objective function emphasizes that income during retirement, rather than wealth at retirement, is the key variable for pensioners. This is not new in the retirement planning literature (Blake et al., 2003; Charupat and Milevsky, 2002; Horneff et al., 2008; Huang et al., 2017; Koijen et al., 2011). The terminal wealth utility cannot measure retirement income and annuity risks properly (Merton, 2014). Although we select income-based utility, the most ideal and practical model would be a consumption-based utility maximization model. For simplicity, we assume that the individual investor spends all of their annuity income to subsidise their consumption after retirement.

Our objective function has limitations. It is implicit in such a function that income-generating financial instruments, such as annuities, are the best financial product for retirees. If annuities are priced with a loading factor to allow for fees and expenses, they will be less appealing than other financial assets. We also assume full annuitization at retirement so that if the bequest to heirs after retirement is an important concern, the objective function would have to be modified accordingly. Similarly, if the individual has an irregular consumption pattern during retirement, or desires a high degree of liquidity, the objective function would also have to be changed to accommodate this.

Available assets. The individual can rebalance their portfolio and buy deferred annuities at regular intervals of length Δt years. There are $N \in \mathbb{N}$ such regular intervals in the retirement planning period $[0, T)$ (i.e. $T = N \cdot \Delta t$) (see Figure 1). Defining $R_{i,t}$ as the accumulated log-return of asset $i \in \{C, B, E\}$ from time $t - \Delta t$ to t, the price $S_{i,t}$ of asset i evolves according to the following:

$$\mathrm{S_{i,t}} = \mathrm{S_{i,t-\Delta t}} \cdot \exp\left(R_{i,t}\right) \qquad \text{for } i \in \{C, B, E\}, \tag{2}$$

where $S_{i,0} = 1$ without loss of generality.

Figure 1 here.

The gross return of the long-term bond fund with a maturity of M years over a holding period of length Δt from time $t - \Delta t$ to t is approximated by

$$\mathrm{R_{B,t}} = M \cdot y(\beta_{t-\Delta t}, M, \lambda) - (M - \Delta t) \cdot y(\beta_t, M - \Delta t, \lambda). \tag{3}$$

The term $y(\beta_t, M, \lambda)$ denotes the M-year spot rate at time t, determined by the Nelson-Siegel term structure model, with parameters β_t and λ, to be specified shortly. Accordingly, the dynamics of the bond fund price is obtained by substituting $R_{i,t}$ from Equation (3) into Equation (2).

The gross return generated by the cash fund is defined by changing the bond maturity M in Equation (3) to Δt. The cash fund return from time $t - \Delta t$ to t, is therefore given by

$$\mathrm{R}_{C,\mathrm{t}} = \Delta \mathrm{t} \cdot \mathrm{y}(\beta_{\mathrm{t}-\Delta \mathrm{t}}, \Delta t, \lambda). \tag{4}$$

Of course, this cash fund return at time t does not depend on the current spot rate $\mathrm{y}(\beta_{\mathrm{t}}, \Delta t, \lambda)$ at time t, but on the past spot rate $\mathrm{y}(\beta_{\mathrm{t}-\Delta \mathrm{t}}, \Delta t, \lambda)$.

For a policyholder aged $\delta + t$ at time t, the fair actuarial price of a deferred annuity contract paying £1 of annual retirement income for a lifetime starting at retirement at time T is

$$\mathrm{S}_{\mathrm{A,t}} = \sum_{s=\mathrm{T-t}}^{\tau-t} {}_{s}P_{\delta+t} \cdot \exp\big(-\mathrm{s} \cdot \mathrm{y}(\beta_{\mathrm{t}}, \mathrm{s}, \lambda)\big). \tag{5}$$

We assume static pricing mortality rates here.

Glide paths with deferred annuities. We approximate a new glide path by averaging stochastic optimal deferred annuity and investment allocations (in percent) over the planning horizon $[0, T)$. Let $G_{i,t}$ be the glide-path strategy (in percent) for asset i at time t and it is given by

$$G_{i,t} = \mathbb{E}_0 \left[S_{i,t} X_{i,t} \Big/ \sum_{j \in \{C,B,E,A\}} S_{j,t} X_{j,t} \right] \qquad \text{for } i \in \{C, B, E, A\}. \tag{6}$$

The glide paths can be constructed with and without deferred annuities.

The glide path strategy is not an optimal investment solution to our retirement planning problem. One alternative could be to construct an optimization problem to search for an optimal glide path $\{G_{i,t}\}$ as one of the decision variables. ~~Constraints to control after-rebalancing asset allocations, however, are to be non-linear equations~~. Finding a global, or close-to-global optimal solution is a hard problem for any currently-available solvers. Our new glide-path strategy can be a good starting point for searching for a better solution.

There are some practical advantages of glide path strategies over the optimal stochastic strategy. First, they can deliver a complex investment strategy to individuals in a way which is easy for individuals to understand. If the strategy is deployed and then updated regularly, then the final retirement income derived for the glide path and optimal stochastic strategies should be the same, since they are identical at time 0.

Financial Modelling and Data

In order to incorporate interest rate uncertainty into the deferred annuity price, a stochastic term structure model is required. We choose the Nelson-Siegel model along with a vector autoregressive (VAR) model for stochastic equity and bond returns.[3] This allows our model to incorporate asset return predictabilities and to use a continuous yield curve for pricing not only the cash and bond funds, but also annuities.

The entire yield curve is determined by a fitted Nelson-Siegel model with three parameters: $\beta_{1,t}$ (level), $\beta_{2,t}$ (slope), and $\beta_{3,t}$ (curvature). The Nelson-Siegel model for the s-year spot rate at time t is as follows:

$$y(\beta_t, s, \lambda) = \beta_{1,t} + (\beta_{2,t} + \beta_{3,t})\left(\frac{1 - e^{-\lambda s}}{\lambda s}\right) - \beta_{3,t} e^{-\lambda s}, \quad (7)$$

where the scaling parameter λ is a constant. Here, $\beta_t = [\beta_{1,t}, \beta_{2,t}, \beta_{3,t}]'$.

To incorporate predictabilities of asset returns and the three parameters in the Nelson-Siegel model, we use a VAR(1) model (for details, see Barberis, 2000; Campbell et al., 2003). In particular, a VAR model combining the interest rate model and equity returns is used, as in Ferstl and Weissensteiner (2011), Pedersen et al. (2013) and Konicz et al. (2016). Our VAR model is given by

$$z_t = \Phi_0 + \Phi_1 z_{t-1} + v_t, \quad (8)$$

[3] Ferstl and Weissensteiner (2011) combine the Nelson-Siegel formulation with the VAR model, which is proposed by Boender et al. (2008).

where $z_t = [r_t, \beta_{1,t}, \beta_{2,t}, \beta_{3,t}]'$. Here, r_t is the monthly log-returns generated by the equity fund. The accumulated return on the equity fund $R_{E,t}$, defined in Equation (2) as the log-return from time $t - \Delta t$ to t, is simply a sum of the monthly log returns. In Equation (8), Φ_0 is a column vector of intercepts, Φ_1 is a 4×4 matrix of the slope coefficients of the VAR model, and v_t is a column vector of *iid* innovations $\sim N(0, \Sigma_z)$, where $\Sigma_z = \mathbb{E}[vv']$.[4]

We use monthly yield curve data calculated by the Bank of England with 0.5 to 25-year spot rates, and returns generated by the FTSE 100 collected from the Bloomberg from January 1993 to December 2013. By minimizing the sum of squared errors between the fitted and historical yield curves, we estimate $\lambda = 0.382$ in Equation (7). Our estimates for Φ_0 and Φ_1 in Equation (8), along with t-statistics, are presented in Table 1. The level of R^2 for the equity equation is low, making it is difficult to confirm that return predictability in the UK equity market exists. The eigenvalues of Φ_1 have moduli less than one, so that the unconditional expected mean and covariance in Equations (9) and (10) exist. Table 2 presents the correlations and standard deviation (multiplied by 100) of the residuals. Table 3 presents the unconditional expected mean μ_{zz} of z_t.

Table 1 here.

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[4] If all eigenvalues of Φ_1 have moduli less than one, the stochastic process in Equation (8) is stable with the unconditional expected mean μ_{zz} and covariance Γ_{zz} of z_t in the steady states:

$$\mu_{zz} = (I - \Phi_1)^{-1}\Phi_0,$$

$$vec(\Gamma_{zz}) = (I - \Phi_1 \otimes \Phi_1)^{-1} vec(\Sigma_z),$$

where I is an identity matrix, the operator $\otimes$ is the Kronecker product, and vec is a vectorisation function, which transforms a $K \times K$ matrix into a $K^2 \times 1$ vector.

Table 2 here.

Table 3 here.

Multi-stage Stochastic Programming Formulation

Scenario generation. An MSP model can be constructed in a nodal form by using state variables generated in a scenario tree. The scenario tree starts at the initial stage from a unique root node which branches out to several "children" nodes at the second time stage. Each of these child nodes themselves branch out to further nodes at the third time stage, and so on. The nodes at the terminal stage are known as leaf nodes. A scenario is the path followed from the root node through descendant nodes to a leaf node. The tree is non-recombining. Some helpful notation pertaining to the scenario tree and the scale of the optimisation problem formulated here is set out in Appendix A.

The optimization problem. The objective function and constraints set out in Equation (1) for the general problem can now be formulated within the scenario tree as a multi-stage stochastic programming problem. The notation transfers in a straightforward way, except that we index by node rather than by time. For example, $X_{i,n}$ refers to the number of units of asset $i \in \{C, B, E, A\}$ held at node n in the scenario tree. We also distinguish between buy and sell decisions, so that $X_{i,n}^{buy}$ is the number of units of asset i to buy at node n and $X_{i,n}^{sell}$ is the number of units of asset i to sell at node n. Recalling that deferred annuities cannot be sold, the decision variable for the individual at node n is therefore $X_n = \left[X_{C,n}^{buy},\ X_{C,n}^{sell},\ X_{B,n}^{buy},\ X_{B,n}^{sell},\ X_{E,n}^{buy},\ X_{E,n}^{sell},\ X_{A,n}^{buy}\right]'$ for $n \in \mathcal{N}$.

The objective function in Equation (1a) is rewritten in a nodal form as follows:

$$\max_{\{X_n, n\in\mathcal{N}\}} \left[\sum_{t\in[T,\tau]} \sum_{n\in\mathcal{N}_T} {}_t p_\delta \; u\big(t, X_{A,n}\big) \, \boldsymbol{pr}_n \right.$$
$$\left. + \sum_{t\in[0,T)} \sum_{n\in\mathcal{N}_{t+\Delta t}} {}_t p_\delta \; {}_{\Delta t} q_{\delta+t} \, u(t+\Delta t, W_n) \, \boldsymbol{pr}_n \right] \tag{9}$$

where it is implicit that summations occur over the time stages in the scenario tree during the planning phase when $t \in [0, T]$.

A cash balance constraint, shown in Equation (10), controls cash inflows and outflows. Below φ_i^s and φ_i^u indicate a percentage selling fee and upfront fee respectively for asset $i \in \{C, B, E, A\}$ and w_0 represents non-random, positive initial wealth.

$$\mathbb{1}_{\{n=n_0\}} w_0 + \mathbb{1}_{\{n\notin\mathcal{N}_T\}} \phi \cdot L_n \cdot \Delta t + \sum_{i\in\{C,B,E\}} X_{i,n}^{sell} \, S_{i,n}(1-\varphi_i^s) = \sum_{i\in\{C,B,E,A\}} X_{i,n}^{buy} \, S_{i,n}(1+\varphi_i^u) \qquad \text{for } n \in \mathcal{N}, \tag{10}$$

An asset inventory constraint shown in Equation (11) tracks the number $X_{i,n}$ of units of asset $i \in \{C, B, E, A\}$ held at node n. Below φ_i^m indicates a percentage management fee for asset i.

$$X_{i,n} = \mathbb{1}_{\{n\notin n_0\}} X_{i,n^-}(1-\varphi_i^m) + X_{i,n}^{buy} - X_{i,n}^{sell} \qquad \text{for } n \in \mathcal{N} \tag{11}$$

Wealth in the retirement plan, which includes cash, bond and equity funds, and excludes purchased deferred annuities, satisfies the following equations:

$$W_n = \sum_{i\in\{C,B,E\}} X_{i,n^-} \, S_{i,n}(1-\varphi_i^m) \quad \text{for } n \in \mathcal{N} \setminus n_0. \tag{12}$$

Variable constraints appear below and complete the multi-stage stochastic programming formulation which is equivalent to the general optimization problem in Equation (1).

$$X_{i,n},\ X_{i,n}^{buy},\ X_{i,n}^{sell} \geq 0 \qquad \text{for } i = \{C, B, E\} \text{ and } n \in \mathcal{N}, \tag{13a}$$

$$X_{A,n},\ X_{A,n}^{buy} \geq 0 \qquad \text{for } n \in \mathcal{N}, \tag{13b}$$

$$X_{i,n} = X_{i,n}^{buy} = 0 \qquad \text{for } i = \{C, B, E\} \text{ and } n \in \mathcal{N}_T. \tag{13c}$$

The terminal condition in Equation (1d) is self-constrained through Equations (10), (11) and (13). The non-negative wealth condition of Equation (1g) is not imposed because it is satisfied in Equation (12) since asset prices are positive and no short-selling is allowed in Equations (11) and (13a). Wealth is initialized at the non-random amount w_0 specified on the l.h.s. of Equation (10).

Following Equation (10) to Equation (13), on every node in the scenario tree, the cash balance, asset inventory and other constraints are set. Finally, we use an interior point solver *MOSEK* to find optimal investment and deferred annuity choices by maximizing the non-linear objective function in Equation (9) subject to the linear constraints in Equations (10) to (13). Hilli et al. (2016), Konicz and Mulvey (2015), and Konicz et al. (2016) use the *MOSEK* for pension asset-liability management and financial planning applications.

Results

Numerical examples. We start with a benchmark case in which a 40-year-old individual ($\delta = 40$) intends to retire at age 65 ($T = 25$). Their goal is to maximize and secure their retirement benefits in nominal terms and to set aside a portion of the portfolio as a bequest in the event of them dying before retirement. In their retirement plan they can invest in a cash fund (maturity $M = 5$ years), a bond fund (maturity $M = 20$ years), an equity fund and in deferred annuities as described earlier. To price the deferred annuity, we use a U.K. mortality table based on 2000-2006 experience.[5]

The individual can rebalance the portfolio and buy deferred annuities every 5 years ($\Delta t = 5$) so there are six stages (five periods) in the scenario tree spanning the 25-year planning horizon. The individual has an initial wealth of $w_0 = £80{,}000$ which represents the initial value of the retirement plan. Annual wage is fixed at £40,000 throughout. Contributions to the retirement plan are £4,000 p.a. ($\phi = 10\ percent$). Because of the incidence of cash flows in our model, the contribution is, in effect, £20,000 every five years in advance ($\phi \cdot L_n \cdot \Delta t = £20{,}000$ for $n \in \mathcal{N} \setminus \mathcal{N}_T$).

In the benchmark case, the individual is a male with risk aversion coefficient $\gamma = 3$, time preference $\rho = 0$ and bequest parameter $\kappa = 0$. For the bond and equity funds, upfront and selling fees are $\varphi_i^u = \varphi_i^s = 0.5\ percent$ for $i \in \{B, E\}$, following Geyer et al. (2009) and Konicz et al. (2014). Expense loadings on annuities are $\varphi_A^u = 3.0$ percent. This is a similar level to the one used by Horneff et al. (2010) and Huang et al. (2017). The cash fund has no fees, and management fees for all assets are ignored. Alternative cases are also considered with different individual preference parameters and transaction costs.

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[5] Institute and Faculty of Actuaries, S1PML/S1PFL - All pensioners (excluding dependants), male/female lives (www.actuaries.org.uk).

We also consider three different fee structures from the benchmark, which we denote case I. A higher fee on bond and equity transactions from 0.5 to 1.0 percent but the same loading as the benchmark case, is referred to as Case II. While Case III uses a higher loading on annuities of 5.0 percent, but the same transaction fee as in the benchmark case. Case IV increases both transaction fees and loading to 1.0 percent and 5.0 percent, respectively.

To control the glide path operation within scenarios we should take transaction fees into account, so that after-rebalancing asset proportions match the target glide path strategy $G_{i,t}$. Let $\tilde{G}_{i,n}$ be the current proportion of asset $i \in \{C, B, E, A\}$ at node n before rebalancing but after making the contribution $\phi \cdot L_n$ into the cash account. We rebalance asset allocations from $\tilde{G}_{i,n}$ to $G_{i,t}$ after fees at every node $n \in \mathcal{N} \setminus \mathcal{N}_T$. The total transaction and loading costs denoting fee_n are calculated by

$$fee_n = \frac{\sum_{i \in \{C,B,E,A\}} sign(i)\left(G_{i,t} - \tilde{G}_{i,n}\right)\varphi_i}{1 + \sum_{\forall i} sign(i) G_{i,t}\varphi_i}(W_n + \phi \cdot L_n), \tag{16}$$

where a function $sign(i)$ returns $+1$ when $\left(G_{i,t} - \tilde{G}_{i,n}\right) > 0$ and -1 when $\left(G_{i,t} - \tilde{G}_{i,n}\right) \leq 0$. If $G_{i,t} = \tilde{G}_{i,n}$ for all $i \in \{C, B, E, A\}$, then no fees occur since rebalancing is not required. Management fees are ignored, but it is not non-trivial. If the current proportion of deferred annuity to wealth plus contribution is greater than that in the glide path, we do not change the deferred annuity allocation ($X_{A,n}$ is equal to X_{A,n^-}) and calculate the total transaction cost in Equation (16), excluding the deferred annuity. In that case, $G_{i,t}$ and $\tilde{G}_{i,n}$ are adjusted by considering only assets $i \in \{C, B, E\}$ as one portfolio and W_n deducted by $S_{A,n}X_{A,n^-}$ in Equation (16).

Traditional glide path strategies. We consider two traditional glide paths. The "Equity-to-Bond" glide-path starts with 80 percent in equities, a proportion that falls by 6 percent points every five years, with the proportion invested in bonds rising commensurately. The "Equity-to-Cash" glide-path also starts with 80 percent in equities, a proportion that falls by 6 percentage points every five years, with the proportion invested in cash rising commensurately. The equity allocations in the traditional glide-path strategies match those suggested by Vanguard (Daga et al., 2016).

Table 4 presents basic statistics for the retirement income secured by the representative individual for a range of the risk aversion parameters, γ, (1, 3, 5, and 8) and the fee structures of cases I, II, III, and IV. The first column indicates the risk aversion parameter and fee structure for each row. The first row for each pair shows the results for the strategy where the investor applies the Equity-to-Bond glide path strategy and purchases an immediate annuity at retirement. The second row shows equivalent statistics, but where the individual implements the Equity-to-Cash glide path, etc.

Table 4 here.

Distributional properties from the Mean to 95th percentile values of the retirement income are identical for the (1, I), (3, II), (5, III), and (8, IV) cases because the strategies do not depend on the individual's risk preference, but are identical and deterministic. The Equity-to-Bond strategy shows a higher mean value, but also has a higher standard deviation (StdDev) than the Equity-to-Cash strategy. This causes a lower expected retirement income per unit of risk (Mean/StdDev) so that we cannot say that the Equity-to-Bond strategy is more efficient than the Equity-to-Cash strategy.

The last column shows the certainty equivalent value of retirement income (CE) to the expected utility of the retirement income at retirement. The Equity-to-Bond strategy produces a higher CE value than the Equity-to-Cash strategy, since investing in the bond with a 20-year maturity is less risky than investing in cash (5-year zero bond) in terms of a price change in one unit of retirement income (i.e. immediate annuity price). In addition, the CE values depends upon each individual's risk preference, although the retirement income has the same distribution in both cases. A glide path strategy can be less valuable to a more risk-averse investor. The CE value of the retirement income ranges between about £4,000 to 6,000 per year. In all four fee cases, I to IV, CE values are lower when transaction fees or expense loading on the annuity are higher. Results with Monte Carlo simulation in Table B.2 also show similarly consistent results.

Optimal Stochastic Strategies. The stochastic investment strategy that results from solving the multi-stage stochastic programming problem, utilizes full information about the realized and expected values of the predictable and time-varying financial variables (for details, see Owadally et al. (2018)). Figure 2 shows the percentiles and average of the total optimally secured retirement income (SRI) generated by employing the MSP approach for the benchmark model, as a function of age. The left panel of Figure 2 shows the total secured retirement income. The right panel shows the extra retirement income gained by purchasing deferred annuities. This benchmark case shows that the optimal strategy to secure retirement income involves buying deferred annuities regularly during the individual's working lifetime, starting fairly early and accelerating in the last years before retirement. The result is robust for other investor profiles. Supplementary results are presented with different risk aversion, time preference and bequest parameters in following sections and in Table B.1.

Figure 2 here.

Average optimal asset allocations over the 25-year planning horizon are presented in Figure 3. The bar graphs show average asset allocations including deferred annuities (DAs) for three different constant relative risk aversion (CRRA) parameters. The chart corresponding to CRRA=3 shows the asset allocation for the benchmark case. The proportion of deferred annuity holding in overall wealth increases on average, as retirement approaches, while bond and equity holdings decline. The fall in equity holdings over time is consistent with findings in the lifetime finance literature and the typical recommendations of financial advisors. Bonds clearly play a significant role in the investor's portfolio. As well as being a relatively "safe asset class", they also represent a partial hedge against the future price changes of deferred annuities.

Figure 3 here.

Bonds are the largest asset holdings among cash, bond and equity funds over the planning horizon, but the hedging demand seems weaker as retirement draws closer. Cairns et al. (2006) and Koijen et al. (2011) find an opposite hedging pattern using a bond, which shows that the hedging demand is stronger as retirement draws closer. They, however, do not consider deferred annuities so that the individual has no means of securing their retirement income in advance.

The optimal allocation to the cash fund increases initially and then declines. The cash account provides liquidity for future deferred annuity purchases, especially when bond and equity prices are low.

Figures 2 and 3 demonstrate that there is a potentially important role for deferred annuities in the accumulation phase. To see just how important, we can compare these results to those where we exclude deferred annuities from the asset allocation choice where we instead only allow the investor to purchase an immediate annuity at retirement.

Table 5 presents statistics equivalent to those in Table 4, but for optimal stochastic strategies with deferred annuity (DA) and with immediate annuity (IA). The first row for each pair of risk aversion level and fee structure shows the results for the strategy where the investor can purchase a deferred annuity. The second row shows equivalent statistics but where the individual is restricted to an immediate annuity only. In each case the difference between the retirement income mean is economically small. However, for each level of γ the individual achieves higher expected retirement income per unit risk (Mean/StdDev) when deferred annuities are available compared to when they are not. The last column, "CE", shows that the certainty equivalent values of retirement income are also higher when deferred annuities are available in all cases. The analysis shows that the availability of deferred annuities provides the individual with not only a higher welfare value but also a more efficient (higher mean-variance) retirement-income distribution.

Table 5 *here.*

Comparing Table 5 with Table 4, we find that the optimal stochastic strategies produce more efficient retirement income distributions (i.e. higher Mean/StdDev values) and higher certainty equivalent value in all cases. The differences in the certainty equivalent values are particularly large, about £26,000 to £34,000 per year, which grow with the risk aversion parameter.

A glide-path approach with deferred annuities. Implementing the optimal stochastic strategy, however, is likely to be too complex for individual investors or their advisors to understand. To this end, we now introduce an investment glide path. The glide path $G_{i,t}$, described in equation (6), defines the proportion of asset i in the retirement plan, including a proportion in a deferred annuity as a function of time. It can be written in the following nodal form:

$$G_{i,t} = \sum_{n \in \mathcal{N}_t} \left[\mathbf{pr}_n \cdot \left(S_{i,n} X_{i,n} \Big/ \sum_{j \in C,B,E,A} S_{j,n} X_{j,n} \right) \right] \text{ for } t \in [0, T) \text{ and } i \in \{C, B, E, A\}.$$

Our numerical results show that this glide path guarantees a better welfare value in terms of certainty equivalent retirement income than traditional glide paths, constant-mix investment, and "$100 - age$" strategies, which are widely used in the retirement planning industry.

Table 6 shows that the outcomes delivered with the introduction of $G_{i,t}$ (in percent) with different transaction costs on bond and equity funds and expense loadings on deferred annuities. All average portfolio allocations are calculated after fees as in Equation (6). When the expense loading increases by 2.0 percent to 5.0 percent (see I and III or II and IV panels), the glide path lowers deferred annuity proportions through the planning horizon. When transaction costs (upfront and selling fees) on bond and equity funds increase by 0.5 percent to 1.0 percent (see I and II or III and IV panels), the new glide path results show higher deferred annuity proportions and lower other proportions in other asset classes through the planning horizon. When the fees on bond and equity funds increase, cash fund allocations increase through the planning period because transaction fees on the cash fund are set to zero.

Table 6 here.

The results in Table 7 with Table 4 and Table 5 demonstrate that the introduction of $G_{i,t}$ produces the most efficient distribution of retirement income and a higher certainty equivalent value of retirement income than traditional glide path strategies. The Mean/StdDev values are much higher than the stochastic optimal strategies. The difference in the certainty equivalent values between the traditional and new glide path is about £3,000 to 12,000 per year. A more risk-averse individual is likely to see a larger difference in the certainty equivalent value. A certainty equivalent gap between the new glide path and the optimal stochastic strategies decreases as the individual becomes more risk averse.

Table 7 here.

It is also helpful to compare our glide path strategies with other deterministic strategies used in practice. We measure two performance metrics: the expected retirement income per unit risk and the certainty equivalent retirement income. This is done for fourteen different strategies and the results are displayed in Figure 4. The fourteen strategies labelled on Figure 4 are as follows:

(A) stochastic optimal strategy with deferred annuities (DA) available including an immediate annuity (IA) at retirement;

(B) stochastic optimal strategy without DAs but including an IA at retirement;

(C) cash only;

(D) bond only;

(E) 70/30 bond/equity;

(F) 50/50 bond/equity;

(G) 30/70 bond/equity;

(H) equity only;

(I) glide path starting from 80/20 equity/bond with equity decreasing and bond increasing by 6 percent point every 5 years;

(J) glide path starting from 80/20 equity/cash with equity decreasing and cash increasing by 6 percent point every 5 years;

(K) $(100 - age)$ percent in equity and the rest in bond;

(L) $(100 - age)$ percent equity and the rest in cash;

(M) glide path strategy with DA available including an IA at retirement; and
(N) glide path strategy without DAs, but including an IA at retirement.

Figure 4 Here.

Figure 4 shows that our glide paths with and without DAs (**(M)** and **(N)** have a better performance than any of the deterministic investment strategies (**(C)** to **(L)**) in terms of the expected retirement income per unit risk. For the certainty equivalent retirement income our new glide path performs better than the typical glide path strategies (**(I)** to **(J)**), but also better than any constant-mix strategies (**(C)** to **(H)**) and "$age - 100$" strategies (**(K)** and **(L)**). The stochastic optimal strategies (**(A)** and **(B)**) clearly show the highest certainty equivalent retirement income. Strategy **(H)** (equity only) has the worst performance among the fourteen strategies, under both performance metrics.

Conclusions

Using multi-stage stochastic programming, we specify an optimization model with the objective function to maximize the expected value of a series of power utility functions of secured retirement income from purchased annuities at retirement and a bequest before retirement. Then we propose and construct new glide paths by averaging optimal stochastic asset allocations which are achieved by solving the retirement planning problem. The multi-stage stochastic programming approach enables us to incorporate sophisticated financial market models and realistic constraints, such as constraints on assets, transaction costs, and taxes.

Numerical results show that our glide path strategies provide the individual with a higher welfare value, and a more efficient investment portfolio than any conventional deterministic strategies, such as constant-mix, traditional glide-path, and " $100 - age$ " investment strategies. The new glide path strategies also make retirement income and wealth more responsive to personal preferences: risk aversion, time impatience and bequest motive.

The welfare value is determined by the certainty equivalent value of the expected utility of secured retirement income. The welfare value gap between our glide paths and the deterministic strategies widens as the investor is more risk-averse. The expected retirement income per unit risk, as a measure of efficiency, is the highest. Ours also shows that the closer-to-optimal certainty equivalent values of the expected utility of bequest over the planning horizon. The existence of a bequest motive results in more allocations to the cash fund and delayed, or lower allocations to deferred annuities. The findings are robust to different fee structures and to the availability of deferred annuities.

The glide path strategy is not an optimal investment solution to our retirement planning problem. However, there is one clear, practical advantage of the glide path strategy over the optimal stochastic strategy: it would be easier to explain the idea to individual investors, since the new glide path strategy is simply a deterministic function of time. Furthermore, if the glide path is updated regularly, the results from the glide path and optimal stochastic strategies will not be very different. This is because their strategies are identical at the time they are updated.

Our glide path strategies have limitations. The model assumes full annuitization at retirement. This implies that annuities are the best income-generating financial instrument. It would have to be modified if the individual wishes to make bequest to heirs after retirement, if the post retirement consumption pattern is irregular, or if a higher degree of liquidity during retirement is required.

We have not taken account of inflation rates which affect consumer prices, housing assets, and wages. Uncertainties in labour income, labour supply, and fiscal policy issues are also ignored. Various types of annuity products, such as index-linked annuities, are not covered. Term assurance and health insurance covering long-term care and critical illnesses are also not considered here. They can affect the optimal retirement planning and our glide path strategy. The results that we have here are influenced by the discretization of state and time in a multi-stage stochastic programming model. In practice, individuals may wish to rebalance their portfolios more often. These limitations will be carefully investigated in our future studies.

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Figures

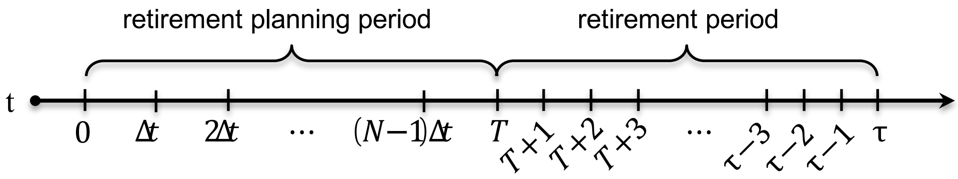

Figure 1. retirement planning and retirement periods from time 0 *to* τ*.*

Note: Maximum life span of an individual is denoted by τ. One increment indicates one year. The time length of Δt is greater than one year.

Source: Author's drawing

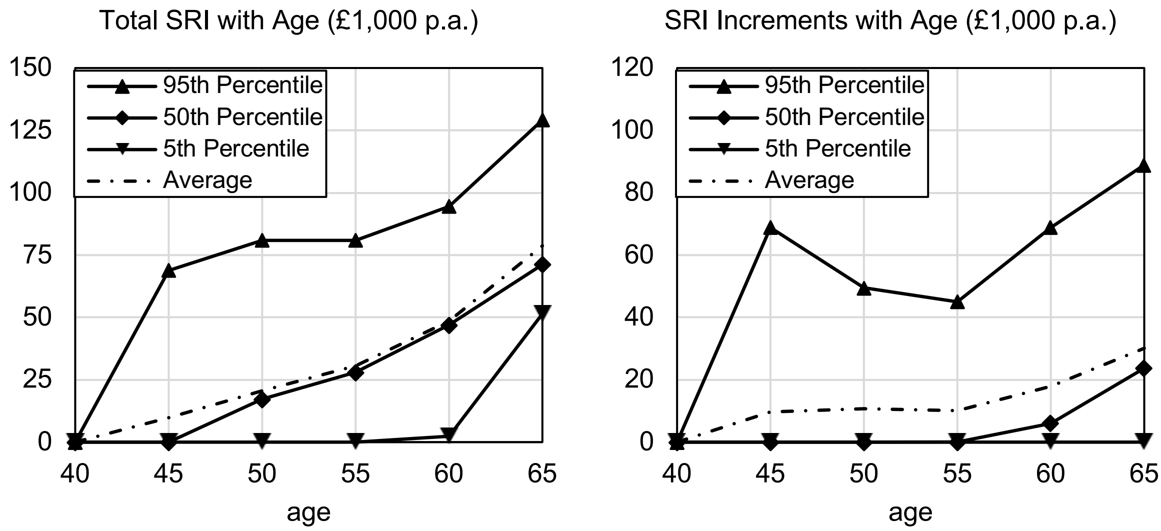

Figure 2. Percentiles and average of optimally secured retirement income (SRI) in total (left) and increments (right).

Note: Constant relative risk aversion, time preference, and bequest parameters are $\gamma = 3.0$, $\rho = 0.0$, and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations

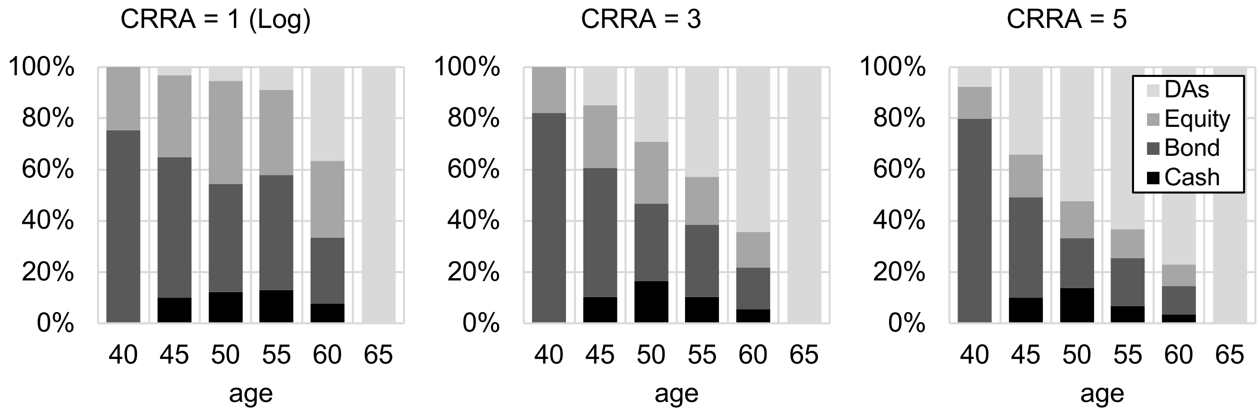

Figure 3. Optimal investment and deferred annuity (DA) allocations on average over the 25-year retirement planning period.

Note: Constant relative risk aversion, time preference, and bequest parameters are $\gamma = \{1.0, 3.0, 5.0\}$, $\rho = 0.0$, and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations

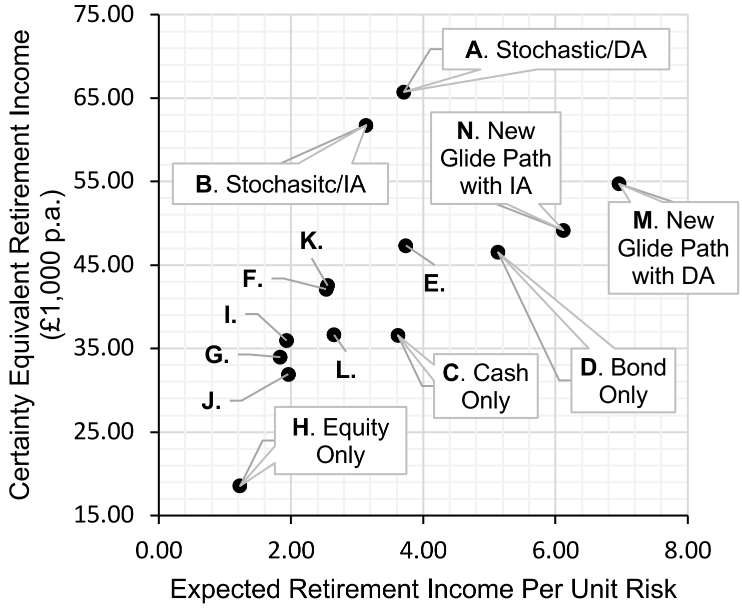

Figure 4. Certainty equivalent retirement income (£1,000 p.a.) and expected retirement income per unit risk for various investment strategies.

Note: Constant relative risk aversion, time preference, and bequest parameters are $\gamma = 5.0$, $\rho = 0.0$, and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations

Tables

Table 1. VAR(1) parameters and t-statistics

	c	A				R^2
		r_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	
r_t	-0.0093	0.0136	0.2446	0.0037	-0.0980	0.0125
t-value	(-0.9961)	(0.2158)	(1.3086)	(0.0266)	(-0.9722)	
$\beta_{1,t}$	0.0070	0.0033	0.8620	-0.0325	0.0229	0.9700
t-value	(4.6116)	(0.3216)	(28.5657)	(-1.467)	(1.4069)	
$\beta_{2,t}$	-0.0044	0.0128	0.0777	1.0008	0.0072	0.9771
t-value	(-4.1225)	(1.7827)	(3.6479)	(63.9633)	(0.6246)	
$\beta_{3,t}$	-0.0024	0.0084	0.0514	0.0206	0.9560	0.9336
t-value	(-1.3018)	(0.6857)	(1.4191)	(0.7742)	(48.9678)	

Note: The scale parameter of a Nelson-Siegel term structure model is set to $\lambda = 0.3820$; t-statistics in parenthesis.

Source: Author's calculations using monthly data of FTSE 100 from Bloomberg and fitted yield curves from the Bank of England from January 1993 to December 2013.

Table 2. Cross correlations and standard deviations of residuals of the VAR(1) model

	r	β_1	β_2	β_3
r	4.0371[a]	-0.0354	0.1487	-0.0180
β_1	-0.0354	0.6518[a]	-0.7944	-0.2002
β_2	0.1487	-0.7944	0.4599[a]	0.0577
β_3	-0.0180	-0.2002	0.0577	0.7821[a]

[a] Standard deviations along the leading diagonal are multiplied by 100.

Source: Author's calculations using monthly data of FTSE 100 from Bloomberg and fitted yield curves from the Bank of England from January 1993 to December 2013.

Table 3. Unconditional expected mean for the steady state of the VAR(1) model

	r	β_1	β_2	β_3
μ_{zz}	0.0040	0.0559	-0.0204	0.0028

Source: Author's calculations using monthly data of FTSE 100 from Bloomberg and fitted yield curves from the Bank of England from January 1993 to December 2013.

Table 4. Secured retirement income (£1,000 p.a.) with traditional glide paths strategies

(γ, Fee)	Strategy	Mean	StdDev	Mean /StdDev	5th Pctl.	50th Pctl.	95th Pctl.	CE[a]
(1, I)	Glide Path (Equity-to-Bond)	56.7035	29.3254	1.9336	25.2069	49.8009	111.6777	50.8701
	Glide Path (Equity-to-Cash)	50.9596	26.0188	1.9586	22.6747	44.9320	99.6784	45.8309
(3, I)	Glide Path (Equity-to-Bond)	56.7035	29.3254	1.9336	25.2069	49.8009	111.6777	42.0999
	Glide Path (Equity-to-Cash)	50.9596	26.0188	1.9586	22.6747	44.9320	99.6784	37.9459
(5, I)	Glide Path (Equity-to-Bond)	56.7035	29.3254	1.9336	25.2069	49.8009	111.6777	35.9490
	Glide Path (Equity-to-Cash)	50.9596	26.0188	1.9586	22.6747	44.9320	99.6784	31.9471
(8, I)	Glide Path (Equity-to-Bond)	56.7035	29.3254	1.9336	25.2069	49.8009	111.6777	29.4541
	Glide Path (Equity-to-Cash)	50.9596	26.0188	1.9586	22.6747	44.9320	99.6784	23.5369
(3, II)	Glide Path (Equity-to-Bond)	56.0250	28.9221	1.9371	24.9271	49.2240	110.2747	41.6249
	Glide Path (Equity-to-Cash)	50.6072	25.8031	1.9613	22.5237	44.6347	98.9885	37.7004
(3, III)	Glide Path (Equity-to-Bond)	55.6235	28.7668	1.9336	24.7268	48.8523	109.5505	41.2980
	Glide Path (Equity-to-Cash)	49.9890	25.5232	1.9586	22.2428	44.0761	97.7797	37.2231
(3, IV)	Glide Path (Equity-to-Bond)	54.9579	28.3712	1.9371	24.4523	48.2864	108.1742	40.8321
	Glide Path (Equity-to-Cash)	49.6432	25.3116	1.9613	22.0947	43.7846	97.1030	36.9823

[a] Certainty equivalent values (£1,000 p.a.) to the expected utility of total secured retirement income at retirement is achieved by solving $u^{-1}(\mathbb{E}[u(T, X_{A,T})]) = CE$; ρ is ignored.

Note: Time preference and bequest coefficients are $\rho = 0.0$ and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations.

Table 5. Secured retirement income (£1,000 p.a.) with optimal stochastic strategies

(γ, Fee)	Strategy	Mean	StdDev	Mean /StdDev	5th Pctl.	50th Pctl.	95th Pctl.	CE[c]
(1, I)	Optimal Stochastic (DA) [a]	84.4278	39.2811	2.1493	42.7146	74.4599	153.8579	77.6503
	Optimal Stochastic (IA) [b]	84.0361	41.3745	2.0311	41.0811	73.4270	159.3199	76.6090
(3, I)	Optimal Stochastic (DA) [a]	78.7150	27.4172	2.8710	51.4306	71.1775	129.1346	69.5943
	Optimal Stochastic (IA) [b]	78.9958	31.2850	2.5250	47.1265	70.9477	137.1415	67.1678
(5, I)	Optimal Stochastic (DA) [a]	73.6425	19.8589	3.7083	54.1586	68.3343	111.1304	65.7513
	Optimal Stochastic (IA) [b]	73.4983	23.4197	3.1383	48.3573	67.8831	118.1316	61.7649
(8, I)	Optimal Stochastic (DA) [a]	68.5278	12.6923	5.3992	55.6949	65.3895	92.1032	63.0439
	Optimal Stochastic (IA) [b]	66.3027	15.5565	4.2621	48.4511	62.9585	94.8632	56.4775
(3, II)	Optimal Stochastic (DA) [a]	77.1447	26.1122	2.9544	50.9577	69.9292	123.9188	68.6372
	Optimal Stochastic (IA) [b]	77.1241	30.0299	2.5682	46.4664	69.6668	132.3535	65.9366
(3, III)	Optimal Stochastic (DA) [a]	77.2157	26.8949	2.8710	50.4510	69.8217	126.6758	68.2687
	Optimal Stochastic (IA) [b]	77.4911	30.6891	2.5250	46.2288	69.5965	134.5292	65.8885
(3, IV)	Optimal Stochastic (DA) [a]	75.6757	25.6162	2.9542	49.9870	68.5980	121.5595	67.3299
	Optimal Stochastic (IA) [b]	75.6550	29.4574	2.5683	45.5814	68.3409	129.8324	64.6807

[a] This strategy is results from the MSP model. Deferred annuities are available at any time before retirement and immediate annuities are available only at retirement.
[b] This strategy is results from the MSP model. Deferred annuities are not available and immediate annuities are only available at retirement.
[c] Certainty equivalent values (£1,000 p.a.) to the expected utility of total secured retirement income at retirement is achieved by solving $u^{-1}\left(\mathbb{E}\left[u\left(T, X_{A,T}\right)\right]\right) = CE$; ρ is ignored.

Note: Time preference and bequest coefficients are $\rho = 0.0$ and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations.

Table 6. New glide path strategies (%) with transaction costs and loadings

	I. transaction fees 0.5%, loadings 3.0%				II. transaction fees 1.0%, loadings 3.0%			
Age	Cash	Bond	Equity	Deferred Annuity	Cash	Bond	Equity	Deferred Annuity
40	0.00	82.11	17.89	0.00	0.00	82.90	17.10	0.00
45	10.23	50.44	24.41	14.92	11.57	48.65	24.49	15.29
50	16.57	30.08	24.18	29.18	16.60	28.59	23.58	31.24
55	10.17	28.33	18.74	42.76	9.86	24.71	18.11	47.32
60	5.49	16.27	13.96	64.28	5.14	14.26	12.87	67.73
65	0.00	0.00	0.00	100.00	0.00	0.00	0.00	100.00
	III. transaction fees 0.5%, loadings 5.0%				IV. transaction fees 1.0%, loadings 5.0%			
Age	Cash	Bond	Equity	Deferred Annuity	Cash	Bond	Equity	Deferred Annuity
40	0.00	82.11	17.89	0.00	0.00	82.90	17.10	0.00
45	10.23	50.54	24.43	14.79	11.57	48.76	24.51	15.16
50	16.59	30.25	24.23	28.93	16.62	28.76	23.63	30.99
55	10.22	28.51	18.84	42.43	9.91	24.89	18.21	46.99
60	5.53	16.37	14.08	64.02	5.18	14.35	12.98	67.49
65	0.00	0.00	0.00	100.00	0.00	0.00	0.00	100.00

Note: The four columns in Panel I to IV indicate averages of the optimal investment and deferred annuity portfolio allocations (%) for the 25-year planning horizon. From the first column, cash, bond, equity funds, and deferred annuities at the given age. The averages are expected values of 15,552 scenarios. Constant relative risk aversion, time preference, and bequest coefficients are $\gamma = 3.0$, $\rho = 0.0$, and $\kappa = 0.0$ respectively. Management fees are ignored.

Source: Author's calculations.

Table 7. Secured retirement income (£1,000 p.a.) with new glide path strategies

(γ, Fee)	Strategy	Mean	StdDev	Mean /StdDev	5th Pctl.	50th Pctl.	95th Pctl.	CE[a]
(1, I)	New Glide Path (DA)	55.2794	14.5486	3.7997	37.1458	52.4395	83.0214	53.5782
	New Glide Path (IA)	54.0513	15.0967	3.5803	35.1125	51.1400	82.9814	52.1801
(3, I)	New Glide Path (DA)	56.2114	10.2498	5.4842	43.0266	54.3734	74.9434	53.8455
	New Glide Path (IA)	52.8994	10.8345	4.8825	38.8616	51.2222	72.9027	50.0871
(5, I)	New Glide Path (DA)	57.1546	8.2102	6.9614	46.2752	55.8850	71.9669	54.7220
	New Glide Path (IA)	52.0794	8.5139	6.1170	40.6307	50.9710	67.4378	49.1856
(8, I)	New Glide Path (DA)	58.1580	6.2038	9.3745	49.8476	57.2504	69.6580	56.0398
	New Glide Path (IA)	51.3827	7.0719	7.2657	41.4260	50.6603	64.0532	48.1055
(3, II)	New Glide Path (DA)	55.8751	9.9935	5.5911	43.0460	54.0264	74.0873	53.6054
	New Glide Path (IA)	52.2684	10.4600	4.9970	38.6516	50.6598	71.5107	49.6012
(3, III)	New Glide Path (DA)	55.1501	10.0393	5.4934	42.2313	53.3500	73.4798	52.8364
	New Glide Path (IA)	51.8918	10.6281	4.8825	38.1214	50.2466	71.5141	49.1331
(3, IV)	New Glide Path (DA)	54.8209	9.7876	5.6011	42.2542	53.0089	72.6823	52.6015
	New Glide Path (IA)	51.2728	10.2608	4.9970	37.9154	49.6949	70.1486	48.6564

[a] Certainty equivalent values (£1,000 p.a.) to the expected utility of total secured retirement income at retirement is achieved by solving $u^{-1}\left(\mathbb{E}\left[u\left(T, X_{A,T}\right)\right]\right) = CE$; ρ is ignored.

Note: Time preference and bequest coefficients are $\rho = 0.0$ and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's calculations.

Appendix A. The MSP Scenario tree

The root node of the scenario tree is denoted by n_0. Let $\mathcal{N}$ be the set of all nodes in the tree, and $\mathcal{N}_t$ be the set of nodes at time t. For our retirement planning problem, time 0 is the first stage and retirement time T is the terminal stage. Thus, $\mathcal{N}_0 = \{n_0\}$ contains the root node only, $\mathcal{N}_T$ is the set of leaf nodes, and $\mathcal{N} = \bigcup_{t\in[0,T]} \mathcal{N}_t$. The unconditional probability that a node, n, occurs is $\mathbf{pr}_n$ and, clearly $\sum_{n\in\mathcal{N}_t} \mathbf{pr}_n = 1$. A node $n \neq n_0$ will branch off from a parent node, denoted by n^-. A node $n \notin \mathcal{N}_T$ will give rise to a set of children nodes, denoted by n^+.

In the operations research literature, scenario trees are generated using three main methods: scenario reduction, state aggregation and moment matching (see Geyer et al., 2010). We choose the moment matching method (Høyland and Wallace, 2001; Klaassen, 2002) for generating scenario trees of accumulated equity returns and three Nelson-Siegel model parameters. The first-period sub-tree has outcomes corresponding to each child node in the set $\mathrm{n_0}^+$. The outcomes for the first period sub-tree are obtained by matching the first four moments of the distributions of state variables. For the second-period sub-trees, the conditional outcomes are obtained by matching the first four moments of the conditional distributions on outcomes of the first-period sub-tree. This procedure is executed sequentially for the third, fourth sub-trees and so on until the final-period sub-trees. By doing so, we ensure that all conditional distribution properties are fully matched throughout the multi-period scenario tree.

The scenario tree that we construct in our multi-stage stochastic programming problem has six stages. The time interval between the stage is Δt, so the stages occur at time 0, Δt, $2\Delta t$,

…, and $T = 5\Delta t$. At each node n, we store the state variables $[R_n, r_n, \beta_{1,n}, \beta_{2,n}, \beta_{3,n}]$ employing the same notation as before except that we index by node n rather than by time. Thus, if node n occurs at time t, R_n denotes the equity log-return over a Δt-long time interval ending at time t (Equation (2)); r_n denotes equity log-return over a month ending at time t (Equation (8)); and $\beta_{1,n}$, $\beta_{2,n}$ and $\beta_{3,n}$ denote the Nelson-Siegel model parameters at time t (Equation (7)). At the root node n_0, the initial state values are set to equal the unconditional expected means in Table 3. In the scenario tree, every non-terminal node branches off to six children nodes. Six outcomes are the minimum required to perfectly match the first four moments of the five state variables.

Validating arbitrage opportunities among the financial assets (cash, bond and equity funds) is dealt with by using the two methods of Klaassen (2002) for two arbitrage types *ex-post* and the method of Geyer et al. (2014) for no-arbitrage bounds *ex-ante*. The detailed step procedures can be found in Owadally et al. (2018).

Since there are six child nodes for every non-terminal node and there are six stages (five periods) there are therefore $6^5 = 7{,}776$ scenarios and $\sum_{j=0}^{5} 6^j = 9{,}331$ nodes. To improve the stability of our results, we aggregate two independently-generated scenario trees, with identical root nodes, into one large scenario tree (see Høyland and Wallace, 2001). This means that the total number of scenarios is 15,552 and the total number of nodes is 18,661.[1]

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[1] Generating each scenario takes about 20 minutes with Matlab by using a parallel loop *parfor* on a HP desktop computer with Intel CPU i7-7700 3.60 Ghz and 32 Gbyte memory.

From the generated outcomes on each node, the asset prices given in Equations (2) to (5) can be rewritten in a nodal form. Recall that any node n in the scenario tree (except for the root node n_0) branches off from a parent node n^- at the previous time stage. The asset price in Equation (2), for example, is transformed into the nodal form simply by replacing t with n and $t - \Delta t$ with n^- as follows:

$$S_{i,n} = S_{i,n^-} \cdot \exp(\mathrm{R_{i,n}}) \qquad \text{for } n \in \mathcal{N} \setminus \mathrm{n_0} \text{ and } i \in \{C, B, E\},$$

where $S_{i,n_0} = 1$. Other pricing formulas are transformed in a similar way.

Appendix B

Table B.1. Average Stochastic Optimal Strategies (%) and Total Secured Retirement Income (£1,000 p.a.)

with Various Risk Aversion (γ), Time Preference (ρ), and Bequest Motive (κ) Parameters

	$\gamma = 1.0, \rho = 0.02, \kappa = 2.0$								$\gamma = 1.0, \rho = 0.02, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	75.60	24.40	0.00	0.00	0.00	0.00	0.00	0.00	75.79	24.21	0.00	0.00	0.00	0.00	0.00
45	9.94	55.88	31.80	2.38	2.13	0.00	0.00	25.38	9.89	57.83	31.85	0.42	0.35	0.00	0.00	0.02
50	12.46	44.17	40.44	2.93	2.61	0.00	0.00	18.30	12.78	45.70	41.02	0.50	0.35	0.00	0.00	0.01
55	14.00	48.58	33.60	3.81	3.26	0.00	0.00	23.32	14.59	50.79	34.16	0.46	0.36	0.00	0.00	0.03
60	10.80	33.74	32.79	22.68	17.94	0.00	0.01	67.02	16.21	43.89	35.41	4.49	3.52	0.00	0.02	23.18
65	0.00	0.00	0.00	100.00	84.48	42.07	74.31	155.72	0.00	0.00	0.00	100.00	84.19	41.23	73.54	158.47

	$\gamma = 3.0, \rho = 0.02, \kappa = 2.0$								$\gamma = 3.0, \rho = 0.02, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	82.06	17.94	0.00	0.00	0.00	0.00	0.00	15.27	67.74	16.99	0.00	0.00	0.00	0.00	0.00
45	10.37	53.41	24.42	11.80	7.58	0.00	0.00	48.03	14.38	60.76	24.86	0.00	0.00	0.00	0.00	0.00
50	16.94	34.45	24.94	23.67	16.19	0.00	15.55	55.67	20.57	52.95	26.48	0.00	0.00	0.00	0.00	0.00
55	11.69	33.46	19.67	35.18	24.81	0.00	24.43	59.25	18.66	58.91	22.43	0.00	0.00	0.00	0.00	0.00

60	8.28	22.24	16.04	53.44	40.33	2.21	40.45	76.35	23.11	48.50	20.73	7.66	5.69	0.00	1.54	18.06
65	0.00	0.00	0.00	100.00	78.95	50.89	71.35	130.96	0.00	0.00	0.00	100.00	78.81	47.15	71.44	134.22
	$\gamma = 5.0, \rho = 0.02, \kappa = 2.0$								$\gamma = 5.0, \rho = 0.02, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	83.09	12.93	3.98	1.69	1.69	1.69	1.69	58.82	32.34	8.84	0.00	0.00	0.00	0.00	0.00
45	10.97	48.81	16.85	23.37	13.06	1.69	10.08	41.10	34.97	47.45	17.57	0.00	0.00	0.00	0.00	0.00
50	15.56	28.55	15.79	40.10	24.70	2.06	24.10	48.65	27.50	54.75	17.75	0.00	0.00	0.00	0.00	0.00
55	9.29	26.40	12.87	51.44	34.10	10.79	34.60	53.32	21.57	63.51	14.92	0.00	0.00	0.00	0.00	0.00
60	6.96	17.66	10.57	64.81	46.47	21.08	46.17	77.22	26.14	49.62	13.84	10.40	7.15	0.00	7.53	15.77
65	0.00	0.00	0.00	100.00	74.20	53.14	68.60	114.30	0.00	0.00	0.00	100.00	72.19	46.17	67.83	111.82
	$\gamma = 8.0, \rho = 0.02, \kappa = 2.0$								$\gamma = 8.0, \rho = 0.02, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	4.28	73.39	6.59	15.74	6.69	6.69	6.69	6.69	77.67	16.84	5.49	0.00	0.00	0.00	0.00	0.00
45	11.75	48.45	8.41	31.40	16.01	6.69	17.07	27.19	66.39	23.89	9.72	0.00	0.00	0.00	0.00	0.00
50	13.40	28.03	9.95	48.62	27.95	14.13	29.17	42.29	54.13	35.67	10.20	0.00	0.00	0.00	0.00	0.00
55	8.62	24.15	8.51	58.73	36.78	21.07	36.79	52.94	29.04	60.93	10.01	0.02	0.01	0.00	0.00	0.00
60	6.80	15.66	7.25	70.29	47.52	30.21	46.98	67.37	31.49	48.69	9.17	10.65	6.42	0.00	6.40	14.28
65	0.00	0.00	0.00	100.00	69.33	53.94	65.98	94.95	0.00	0.00	0.00	100.00	62.10	41.93	59.30	91.41
	$\gamma = 1.0, \rho = 0.04, \kappa = 2.0$								$\gamma = 1.0, \rho = 0.04, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	75.61	24.39	0.00	0.00	0.00	0.00	0.00	0.00	75.79	24.21	0.00	0.00	0.00	0.00	0.00

45	9.94	56.20	31.80	2.06	1.83	0.00	0.00	20.64	9.92	58.19	31.83	0.06	0.05	0.00	0.00	0.00
50	12.50	44.57	40.55	2.38	2.05	0.00	0.00	12.66	12.95	45.93	41.04	0.08	0.05	0.00	0.00	0.00
55	14.18	49.25	33.69	2.88	2.44	0.00	0.00	17.48	14.71	51.01	34.21	0.08	0.05	0.00	0.00	0.00
60	11.26	34.82	33.16	20.77	16.37	0.00	0.01	63.67	16.95	44.90	35.60	2.55	2.01	0.00	0.01	16.00
65	0.00	0.00	0.00	100.00	84.47	42.05	74.30	156.19	0.00	0.00	0.00	100.00	84.12	41.15	73.45	158.84
	$\gamma = 3.0, \rho = 0.04, \kappa = 2.0$								$\gamma = 3.0, \rho = 0.04, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	82.05	17.95	0.00	0.00	0.00	0.00	0.00	21.15	62.33	16.52	0.00	0.00	0.00	0.00	0.00
45	10.41	54.21	24.42	10.95	7.05	0.00	0.00	44.81	15.73	59.63	24.64	0.00	0.00	0.00	0.00	0.00
50	17.09	35.50	25.12	22.29	15.23	0.00	14.44	52.03	20.84	52.62	26.54	0.00	0.00	0.00	0.00	0.00
55	12.00	34.42	19.88	33.70	23.74	0.00	22.72	55.27	18.77	58.74	22.49	0.00	0.00	0.00	0.00	0.00
60	8.58	22.83	16.26	52.33	39.49	2.20	39.80	74.65	23.84	49.49	20.83	5.84	4.32	0.00	0.01	14.77
65	0.00	0.00	0.00	100.00	79.02	50.78	71.39	131.60	0.00	0.00	0.00	100.00	78.62	46.99	71.36	133.90
	$\gamma = 5.0, \rho = 0.04, \kappa = 2.0$								$\gamma = 5.0, \rho = 0.04, \kappa = 10.0$							
Age	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	0.00	85.04	12.95	2.01	0.86	0.86	0.86	0.86	60.42	30.76	8.81	0.00	0.00	0.00	0.00	0.00
45	11.62	50.94	16.69	20.76	11.78	0.86	9.00	38.52	38.08	44.66	17.26	0.00	0.00	0.00	0.00	0.00
50	16.10	29.91	15.86	38.14	23.55	1.59	22.77	46.25	29.54	52.63	17.83	0.00	0.00	0.00	0.00	0.00
55	9.64	27.24	13.04	50.08	33.19	10.25	33.86	52.54	22.04	62.91	15.05	0.00	0.00	0.00	0.00	0.00
60	7.19	18.10	10.68	64.03	45.94	20.76	45.56	76.80	26.94	50.50	13.90	8.66	5.90	0.00	5.95	13.56
65	0.00	0.00	0.00	100.00	74.28	52.97	68.67	114.84	0.00	0.00	0.00	100.00	71.93	45.62	67.55	111.84
Age	$\gamma = 8.0, \rho = 0.04, \kappa = 2.0$								$\gamma = 8.0, \rho = 0.04, \kappa = 10.0$							

	Avg. Stochastic Optimal Strategy				Total Secured Retirement Income				Avg. Stochastic Optimal Strategy				Total Secured Retirement Income			
	Cash	Bond	Equity	DA	Avg.	5th	50th	95th	Cash	Bond	Equity	DA	Avg.	5th	50th	95th
40	5.07	77.29	6.04	11.60	4.93	4.93	4.93	4.93	77.93	16.56	5.51	0.00	0.00	0.00	0.00	0.00
45	12.62	51.17	7.89	28.33	14.58	4.93	15.75	25.08	67.06	23.20	9.74	0.00	0.00	0.00	0.00	0.00
50	13.92	29.08	9.96	47.04	27.02	12.97	28.45	41.37	57.08	32.69	10.22	0.01	0.01	0.00	0.00	0.00
55	8.91	24.81	8.55	57.73	36.11	19.62	36.26	52.23	31.31	58.65	10.02	0.02	0.02	0.00	0.00	0.01
60	6.96	15.94	7.28	69.82	47.17	29.87	46.77	66.67	32.25	49.22	9.20	9.33	5.56	0.00	5.41	13.08
65	0.00	0.00	0.00	100.00	69.29	53.69	65.98	94.69	0.00	0.00	0.00	100.00	61.78	41.62	58.94	91.26

Note: Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored.

Source: Author's Calculation

Table B.2. Monte Carlo Simulation Results (£1,000 p.a.) with different risk aversion and fee structures

(γ, Fee[a])	Strategy	Mean	StdDev	Mean /StdDev	5th Pctl.	50th Pctl.	95th Pctl.	CE[b]
(1, I)	New Glide Path (DA)	55.2354	14.3625	3.8458	36.5790	52.8877	81.8111	53.5514
	New Glide Path (IA)	54.0105	14.9657	3.6090	34.7661	51.4905	81.7533	52.1548
	Glide Path (Equity-to-Bond)	56.6303	28.8644	1.9619	24.8626	49.9473	110.7636	50.8668
	Glide Path (Equity-to-Cash)	50.9162	25.6795	1.9828	22.5801	45.0195	98.9467	45.8338
(3, I)	New Glide Path (DA)	56.1819	9.9666	5.6370	42.2370	54.9670	74.2333	53.7750
	New Glide Path (IA)	52.8627	10.5864	4.9934	38.3043	51.4654	72.1647	50.0278
	Glide Path (Equity-to-Bond)	56.6303	28.8644	1.9619	24.8626	49.9473	110.7636	41.9321
	Glide Path (Equity-to-Cash)	50.9162	25.6795	1.9828	22.5801	45.0195	98.9467	37.9330
(5, I)	New Glide Path (DA)	57.1334	7.9538	7.1831	45.5161	56.3952	71.2707	54.5818
	New Glide Path (IA)	52.0478	8.2361	6.3195	40.1720	51.2051	66.8108	49.0862
	Glide Path (Equity-to-Bond)	56.6303	28.8644	1.9619	24.8626	49.9473	110.7636	35.4050
	Glide Path (Equity-to-Cash)	50.9162	25.6795	1.9828	22.5801	45.0195	98.9467	32.1310
(8, I)	New Glide Path (DA)	58.1401	6.0280	9.6450	49.2704	57.6070	68.8223	55.8784
	New Glide Path (IA)	51.3565	6.8259	7.5238	41.1714	50.8061	63.4013	48.0088
	Glide Path (Equity-to-Bond)	56.6303	28.8644	1.9619	24.8626	49.9473	110.7636	28.4282
	Glide Path (Equity-to-Cash)	50.9162	25.6795	1.9828	22.5801	45.0195	98.9467	25.8880
(3, II)	New Glide Path (DA)	55.8462	9.7047	5.7545	42.2138	54.6927	73.4043	53.5415
	New Glide Path (IA)	52.2352	10.2227	5.1097	38.1173	50.9105	70.8437	49.5485
	Glide Path (Equity-to-Bond)	55.9526	28.4682	1.9654	24.5905	49.3711	109.3376	41.4596
	Glide Path (Equity-to-Cash)	50.5617	25.4659	1.9855	22.4375	44.7213	98.1971	37.6860
(3, III)	New Glide Path (DA)	55.1213	9.7612	5.6470	41.4591	53.9340	72.7986	52.7673
	New Glide Path (IA)	51.8558	10.3848	4.9934	37.5747	50.4851	70.7902	49.0749
	Glide Path (Equity-to-Bond)	55.5516	28.3146	1.9619	24.3890	48.9959	108.6538	41.1334
	Glide Path (Equity-to-Cash)	49.9463	25.1903	1.9828	22.1500	44.1620	97.0620	37.2105

(3, IV)	New Glide Path (DA)	54.7926	9.5040	5.7652	41.4375	53.6653	71.9870	52.5389
	New Glide Path (IA)	51.2402	10.0280	5.1097	37.3913	49.9408	69.4943	48.6047
	Glide Path (Equity-to-Bond)	54.8869	27.9260	1.9654	24.1221	48.4307	107.2550	40.6699
	Glide Path (Equity-to-Cash)	49.5986	24.9809	1.9855	22.0101	43.8695	96.3267	36.9682

[a] Fee structure labelling refers to Table 6.
[b] Certainty equivalent values to the expected utility of total secured retirement income at retirement is achieved by solving $u^{-1}\left(\mathbb{E}\left[u\left(T, X_{A,T}\right)\right]\right) = CE$; ρ is ignored.

Note: For the upper panel, time preference, and bequest coefficients are $\rho = 0.0$ and $\kappa = 0.0$ respectively. Upfront and selling fees are 0.0% for the cash fund and 0.5% for the bond and equity funds. Expense loadings on annuities are 3.0%. Management fees are ignored. For the lower panel, constant risk aversion, time preference, and bequest coefficients are $\gamma = 3.0$, $\rho = 0.0$ and $\kappa = 0.0$ respectively.

Source: Author's calculations.